

# HIGHER ORDER POISSON KERNELS AND $L^p$ POLYHARMONIC BOUNDARY VALUE PROBLEMS IN LIPSCHITZ DOMAINS

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**ABSTRACT.** In this article, we introduce higher order conjugate Poisson and Poisson kernels, which are higher order analogues of the classical conjugate Poisson and Poisson kernels, as well as the polyharmonic fundamental solutions, and define multi-layer potentials in terms of Poisson field and the polyharmonic fundamental solutions, in which the former formed by the higher order conjugate Poisson and Poisson kernels. Then by the multi-layer potentials, we solve three classes of boundary value problems (i.e., Dirichlet, Neumann and regularity problems) with  $L^p$  boundary data for polyharmonic equations in Lipschitz domains and give integral representation (or potential) solutions of these problems.

## 1. INTRODUCTION

Let  $D$  be a Lipschitz graphic domain or bounded Lipschitz domain in  $\mathbb{R}^{n+1}$ ,  $n \geq 2$ . In this work, we will resolve the following boundary value problems (simply, BVPs) for polyharmonic functions in  $D$  with  $L^p$  boundary data:

**Dirichlet problem:**

$$(1.1) \quad \begin{cases} \Delta^m u = 0, & \text{in } D, \\ \Delta^j u = f_j, & \text{on } \partial D, \\ (u - M_1 \tilde{f}_0) \in L^p(D) \end{cases}$$

with  $\|u - M_1 \tilde{f}_0\|_{L^p(\partial D)} \leq C \sum_{j=1}^{m-1} \|f_j\|_{L^p(\partial D, w d\sigma)}$ , where  $\Delta$  is the Laplacian,  $f_0 \in L^p(\partial D)$ ,  $f_j \in L^p(\partial D, w d\sigma)$ ,  $1 \leq j \leq m-1$  for some  $p \in (1, \infty)$  and some certain weight functions  $w$  on  $\partial D$  (if  $D$  is bounded,  $w \equiv 1$ ),  $d\sigma$  is the area measure of  $\partial D$ ,  $\tilde{f}_0$  is related to all the boundary data  $f_j$ ,

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$0 \leq j \leq m-1$ ,  $M_1$  is the classical double layer potential operator, and the constant  $C$  depends only on  $m, n, p$  and  $D$ .

**Neumann problem:**

$$(1.2) \quad \begin{cases} \Delta^m u = 0, & \text{in } D, \\ \frac{\partial}{\partial N} \Delta^j u = g_j, & \text{on } \partial D, \\ \nabla(u - \mathcal{M}_1 \tilde{g}_0) \in L^p(D) \end{cases}$$

with  $\|\nabla(u - \mathcal{M}_1 \tilde{g}_0)\|_{L^p(\partial D)} \leq C \sum_{j=1}^{m-1} \|g_j\|_{L^p(\partial D, w d\sigma)}$ , where  $\Delta$  is the Laplacian,  $\nabla$  is the gradient operator,  $\frac{\partial}{\partial N}$  denotes the outward normal derivative,  $g_0 \in L^p(\partial D)$ ,  $g_j \in L^p(\partial D, w d\sigma)$ ,  $1 \leq j \leq m-1$  for some  $p \in (1, \infty)$  and some certain weight functions  $w$  on  $\partial D$  (if  $D$  is bounded,  $w \equiv 1$ , and  $g_{m-1}$  has mean value zero, i.e.  $\int_{\partial D} g_{m-1} d\sigma = 0$ ),  $d\sigma$  is the area measure of  $\partial D$ ,  $\tilde{g}_0$  is related to all the boundary data  $g_j$ ,  $0 \leq j \leq m-1$ ,  $\mathcal{M}_1$  is the classical single layer potential operator, and the constant  $C$  depends only on  $m, n, p$  and  $D$ .

**Regularity problem:**

$$(1.3) \quad \begin{cases} \Delta^m u = 0, & \text{in } D, \\ \Delta^j u = h_j, & \text{on } \partial D, \\ \nabla(u - \mathcal{M}_1 \tilde{h}_0) \in L^p(D) \end{cases}$$

with  $\|\nabla(u - \mathcal{M}_1 \tilde{h}_0)\|_{L^p(D)} \leq C \sum_{j=1}^{m-1} \|h_j\|_{L_1^p(\partial D, w d\sigma)}$ , where  $\Delta$  is the Laplacian,  $\nabla$  is the gradient operator,  $h_0 \in L_1^p(\partial D)$ ,  $h_j \in L_1^p(\partial D, w d\sigma)$ ,  $0 \leq j \leq m-1$  for some  $p \in (1, \infty)$  and some certain weight functions  $w$  on  $\partial D$  (if  $D$  is bounded,  $w \equiv 1$ ),  $d\sigma$  is the area measure of  $\partial D$ ,  $\tilde{h}_0$  is related to all the boundary data  $h_j$ ,  $0 \leq j \leq m-1$ ,  $\mathcal{M}_1$  is the classical single layer potential operator, and the constant  $C$  depends only on  $m, n, p$  and  $D$ .

Moreover, as the classical results for the Laplace's equation, in the case of bounded Lipschitz domains, we also have the following estimates of the solutions:

- $\|M(u)\|_{L^p(\partial D)} \leq C \sum_{j=0}^{m-1} \|f_j\|_{L^p(\partial D)}$  for the polyharmonic Dirichlet problem (simply, PHD problem);
- $\|M(\nabla u)\|_{L^p(\partial D)} \leq C \sum_{j=0}^{m-1} \|g_j\|_{L^p(\partial D)}$  and  $\|u\|_{L^p(D)} \leq C \sum_{j=0}^{m-1} \|g_j\|_{L^p(\partial D)}$  for the polyharmonic Neumann problem (simply, PHN problem);
- $\|M(\nabla u)\|_{L^p(\partial D)} \leq C \sum_{j=0}^{m-1} \|h_j\|_{L_1^p(\partial D)}$  and  $\|u\|_{L^p(D)} \leq C \sum_{j=0}^{m-1} \|h_j\|_{L_1^p(\partial D)}$  for the polyharmonic regularity problem (simply, PHR problem),

where  $M(u)$  and  $M(\nabla u)$  are respectively the non-tangential maximal functions of  $u$  and  $\nabla u$ , which is defined by

$$(1.4) \quad M(F)(Q) = \sup_{X \in \Gamma_\gamma(Q)} |F(X)|, \text{ for } Q \in \partial D,$$

where  $\Gamma_\gamma(Q)$  is the non-tangential approach region, viz.,

$$(1.5) \quad \Gamma_\gamma(Q) = \{X \in D : |X - Q| < \gamma \text{dist}(X, \partial D)\}$$

in which  $\gamma > 1$ . It is worthy to note that the non-tangential maximal functions  $M(F)$ , and the non-tangential limits  $\lim_{\substack{X \rightarrow P \\ X \in \Gamma_\gamma(P), P \in \partial D}} F(X)$  throughout this article, are

defined for all  $\gamma > 0$ , so we always elide the subscript  $\gamma$  in proper places and denote  $\Gamma_\gamma(\cdot)$  only by  $\Gamma(\cdot)$ . It is also clear that all the boundary data in BVPs (1.1)-(1.3)

are non-tangential. Throughout this paper, all the spaces  $L_1^p(\partial D, wd\sigma)$  have the same sense as the case of Laplace equation (for the details, see [11, 12, 53]).

Since the late of 1970s, there was a great deal of activity on the study of boundary value problems for partial differential equations in Lipschitz domains. The first breakthrough was due to Dahlberg. In 1977, through a careful analysis of the Poisson kernel of a Lipschitz domain  $D$  with which given, he showed that there exists an  $\varepsilon > 0$  depending only on the geometry of  $D$  such that the Dirichlet problem is solvable for the data in  $L^p(\partial D, d\sigma)$ ,  $2 - \varepsilon < p < \infty$  (see [8–10]). In 1978, Fabes, Jodeit and Riviere used Calderón theorem on the boundedness of the Cauchy integrals on Lipschitz curves for a special case [6], to extend the classical method of layer potentials to  $C^1$  domains. Thus they resolved the Dirichlet and Neumann problems for Laplace's equation, with  $L^p(\partial D, d\sigma)$  and optimal estimates, for  $C^1$  domains [23]. In 1979, by using an identity due to Rellich, Jerison and Kenig gave a simple proof of Dahlberg's results and resolved the Neumann problem on Lipschitz domains, with  $L^2(\partial D, d\sigma)$  and optimal estimates [30–32]. In 1981, Coifman, McIntosh and Meyer established their deep theorem on the boundedness of the Cauchy integral on any Lipschitz curve for general case [7]. Using Coifman-McIntosh-Meyer theorem and Rellich type formula, in 1982, Verchota extended the  $C^1$  results of Fabes, Jodeit and Riviere to the Dirichlet problem in  $L^2(\partial D, d\sigma)$  for Laplace's equation in Lipschitz domains in terms of the method of layer potentials [53]. It was due to Dahlberg and Kenig to resolve the Neumann problem in  $L^p(\partial D, d\sigma)$  for Laplace's equation in Lipschitz domains in 1987 [12]. Thereafter, the technique of layer potentials became an overwhelming method in the study of BVPs in  $C^1$  and Lipschitz domains of Euclidean spaces or Riemann manifolds, with various boundary data, including the Hölder continuous,  $L^p$ , Hardy, Besov, Sobolev types etc.. The BVP types included Dirichlet, Neumann, Robin and mixed problems for elliptic equations and elliptic systems [11–15, 34–41, 43–48, 53–56]. Although there were some works for higher order equations (principally, polyharmonic [13, 38, 47, 55]), however, the most were second order elliptic boundary value problems [33, 37] and biharmonic boundary value problems [14, 36, 44–46, 54, 56].

In this paper, we introduce higher order conjugate Poisson and Poisson kernels, which are higher order analogues of the classical conjugate Poisson and Poisson kernels, as well as the polyharmonic fundamental solutions, and define multi-layer potentials in terms of Poisson field and the polyharmonic fundamental solutions, in which the former formed by the higher order conjugate Poisson and Poisson kernels. Then by the multi-layer potentials, we solve three classes of boundary value problems (i.e., Dirichlet, Neumann and regularity problems) with  $L^p$  boundary data for polyharmonic equations in Lipschitz domains and give integral representation (or potential) solutions of these problems. That is, combining with the known second order results of Dahlberg, Kenig and Verchota etc., we resolve the higher order elliptic boundary value problems (1.1)-(1.3) in Lipschitz domains.

## 2. HIGHER ORDER CONJUGATE POISSON AND POISSON KERNELS

It is well-known that the conjugate Poisson and Poisson kernels in  $\mathbb{R}^{n+1}$  can be unifiedly denoted as the following form up to a different constant (see [51])

$$(2.1) \quad \mathcal{P}_j(x) = C_n \frac{x_j}{|x|^{n+1}},$$

where  $x = (x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$ ,  $1 \leq j \leq n+1$  and

$$(2.2) \quad C_n = \frac{1}{\omega_n} = \frac{\Gamma(\frac{n+1}{2})}{2\pi^{\frac{n+1}{2}}},$$

in which  $\omega_n$  is the surface area of the unit sphere  $S^n$  in  $\mathbb{R}^{n+1}$ .

In what follows, we will introduce higher order conjugate Poisson and Poisson kernels in terms of  $\mathcal{P}_j$ .

**Lemma 2.1.** *Let  $x = (x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$ , then for any  $s \in \mathbb{R}$  and  $1 \leq j \leq n+1$ ,*

$$(2.3) \quad \Delta(x_j |x|^s) = s(s+n+1)x_j |x|^{s-2}$$

and

$$(2.4) \quad \Delta(x_j |x|^s \log |x|) = s(s+n+1)x_j |x|^{s-2} \log |x| + (2s+n+1)x_j |x|^{s-2},$$

where  $\Delta = \sum_{k=1}^{n+1} \frac{\partial^2}{\partial x_k^2}$  and  $|x| = \sqrt{x_1^2 + \dots + x_{n+1}^2}$ .

*Proof.* It is the same as in [22]. □

Denote that

$$(2.5) \quad \alpha_s = s(s+n+1)$$

for any  $s \in \mathbb{R}$ . Thus, when  $s \neq 0$ , we can rewrite (2.3) and (2.4) as follows:

$$(2.6) \quad \Delta \left( \frac{1}{\alpha_s} x_j |x|^s \right) = x_j |x|^{s-2}$$

and

$$(2.7) \quad \Delta \left( \frac{1}{\alpha_s} x_j |x|^s \log |x| \right) = x_j |x|^{s-2} \log |x| + \left( \frac{1}{s} + \frac{1}{s+n+1} \right) x_j |x|^{s-2}.$$

As a convention, we take that  $\alpha_0 = 1$ . Moreover, we also have

$$(2.8) \quad \Delta \left( \frac{1}{n+1} x_j \log |x| \right) = x_j |x|^{-2}.$$

**Lemma 2.2.** *Suppose that  $x = (x_1, x_2, \dots, x_{n+1}), v = (v_1, v_2, \dots, v_{n+1}) \in \mathbb{R}^{n+1}$ . Let*

$$(2.9) \quad D_1^{(j)}(x, v) = -\mathcal{P}_j(x - v).$$

For  $m \in \mathbb{N}$  and  $m \geq 2$ , define

$$(2.10) \quad D_m^{(j)}(x, v) = \frac{c_n}{\beta_1 \beta_2 \dots \beta_{m-1}} (x_j - v_j) |x - v|^{2m-(n+3)}$$

if  $n$  is even, and

$$(2.11) \quad D_m^{(j)}(x, v) = \begin{cases} \frac{c_n}{\beta_1 \beta_2 \dots \beta_{m-1}} (x_j - v_j) |x - v|^{2m-(n+3)}, & m \leq \frac{n+1}{2}, \\ \frac{c_n}{(n+1)\beta_1 \beta_2 \dots \beta_{\frac{n+1}{2}-1} \alpha_2 \alpha_4 \dots \alpha_{2m-n-3}} (x_j - v_j) |x - v|^{2m-(n+3)} \\ \times \left[ \log |x - v| - \sum_{t=1}^{m-\frac{n+3}{2}} \left( \frac{1}{2t} + \frac{1}{2t+n+1} \right) \right], & m \geq \frac{n+3}{2} \end{cases}$$

if  $n$  is odd, where

$$(2.12) \quad \beta_k = \alpha_{2k-n-1}, \quad k = 1, 2, \dots, m-1,$$

$\alpha_s$  is given by (2.5) and  $c_n = -C_n$ ,  $C_n$  is given by (2.2). Then

$$(2.13) \quad \Delta D_1^{(j)}(x, v) = 0 \text{ and } \Delta D_m^{(j)}(x, v) = D_{m-1}^{(j)}(x, v), \quad m \geq 2.$$

*Proof.* By direct calculations, it immediately follows from (2.6)-(2.8).  $\square$

In the following, we need to introduce ultraspherical polynomials [1, 52],  $P_l^{(\lambda)}$  and  $Q_l^{(\lambda)}$ , which can be respectively defined by the generating functions

$$(2.14) \quad (1 - 2r\xi + r^2)^{-\lambda} = \sum_{l=0}^{\infty} P_l^{(\lambda)}(\xi) r^l$$

and

$$(2.15) \quad (1 - 2r\xi + r^2)^{-\lambda} \log(1 - 2r\xi + r^2) = \sum_{l=0}^{\infty} Q_l^{(\lambda)}(\xi) r^l,$$

where  $\lambda \neq 0$ ,  $0 \leq |r| < 1$  and  $|\xi| \leq 1$ .  $P_l^{(\lambda)}$  and  $Q_l^{(\lambda)}$  have the following explicit expressions:

$$(2.16) \quad \begin{aligned} P_l^{(\lambda)}(\xi) &= \frac{1}{l!} \left\{ \frac{d^l}{dr^l} [(1 - 2r\xi + r^2)^{-\lambda}] \right\}_{r=0} \\ &= \sum_{j=0}^{\lfloor \frac{l}{2} \rfloor} (-1)^j \frac{\Gamma(l - j + \lambda)}{\Gamma(\lambda) j! (l - 2j)!} (2\xi)^{l-2j} \end{aligned}$$

and

$$(2.17) \quad \begin{aligned} Q_l^{(\lambda)}(\xi) &= -\frac{d}{d\lambda} [P_l^{(\lambda)}(\xi)] \\ &= \sum_{j=0}^{\lfloor \frac{l}{2} \rfloor} \sum_{k=0}^{l-j-1} (-1)^{j+1} \frac{\Gamma(l - j + \lambda)}{(\lambda + k) \Gamma(\lambda) j! (l - 2j)!} (2\xi)^{l-2j}, \end{aligned}$$

where  $\lfloor \frac{l}{2} \rfloor$  denotes the integer part of  $\frac{l}{2}$ . If necessary, for some special values of  $\lambda$ , say  $\lambda = \lambda_0$ , the above expressions may be extended and interpreted as limits for  $\lambda \rightarrow \lambda_0$  (for example,  $\lambda$  is a non-positive integer). Some other properties of the ultraspherical polynomials can be also found in [1, 52].

For sufficiently large  $|v| \geq |x|$  and any real numbers  $\lambda \neq 0$ ,

$$(2.18) \quad \begin{aligned} |x - v|^{-2\lambda} &= (|v|^2 - 2x \cdot v + |x|^2)^{-\lambda} \\ &= |v|^{-2\lambda} \left[ 1 - 2 \frac{|x|}{|v|} \left( \frac{x}{|x|} \cdot \frac{v}{|v|} \right) + \frac{|x|^2}{|v|^2} \right]^{-\lambda} \\ &= |v|^{-2\lambda} \sum_{l=0}^{\infty} P_l^{(\lambda)}(x_{S^n} \cdot v_{S^n}) \left( \frac{|x|}{|v|} \right)^l \\ &= \sum_{l=0}^{\infty} |x|^l P_l^{(\lambda)}(x_{S^n} \cdot v_{S^n}) |v|^{-(l+2\lambda)}, \end{aligned}$$

where  $x_{S^n} = \frac{x}{|x|}$  and  $v_{S^n} = \frac{v}{|v|}$ . Obviously,  $x_{S^n}, v_{S^n} \in S^n$ .

Similarly, we have

(2.19)

$$\begin{aligned}
& |x - v|^{-2\lambda} \log |x - v| \\
&= |x - v|^{-2\lambda} \left[ \frac{1}{2} \log \frac{|x - v|^2}{|v|^2} + \log |v| \right] \\
&= (|v|^2 - 2x \cdot v + |x|^2)^{-\lambda} \left[ \frac{1}{2} \log \frac{|v|^2 - 2x \cdot v + |x|^2}{|v|^2} + \log |v| \right] \\
&= |v|^{-2\lambda} \left[ 1 - 2 \frac{|x|}{|v|} \left( \frac{x}{|x|} \cdot \frac{v}{|v|} \right) + \frac{|x|^2}{|v|^2} \right]^{-\lambda} \left\{ \frac{1}{2} \log \left[ 1 - 2 \frac{|x|}{|v|} \left( \frac{x}{|x|} \cdot \frac{v}{|v|} \right) + \frac{|x|^2}{|v|^2} \right] \right. \\
&\quad \left. + \log |v| \right\} \\
&= \frac{1}{2} |v|^{-2\lambda} \sum_{l=0}^{\infty} Q_l^{(\lambda)}(x_{S^n} \cdot v_{S^n}) \left( \frac{|x|}{|v|} \right)^l + |v|^{-2\lambda} \log |v| \sum_{l=0}^{\infty} P_l^{(\lambda)}(x_{S^n} \cdot v_{S^n}) \left( \frac{|x|}{|v|} \right)^l \\
&= \frac{1}{2} \sum_{l=0}^{\infty} |x|^l Q_l^{(\lambda)}(x_{S^n} \cdot v_{S^n}) |v|^{-(l+2\lambda)} + \sum_{l=0}^{\infty} |x|^l \log |v| P_l^{(\lambda)}(x_{S^n} \cdot v_{S^n}) |v|^{-(l+2\lambda)}.
\end{aligned}$$

**Definition 2.3.** Let  $f$  be a continuous function defined in  $\mathbb{R}^{n+1}$  that can be expanded as

$$(2.20) \quad f(\zeta) = \sum_{k=-\infty}^m c_k(\zeta) |\zeta|^k$$

for sufficiently large  $|\zeta|$ , where the integer  $m \geq -(n+1)$  and the coefficient functions  $c_k(\zeta)$  are continuous in  $\mathbb{R}^{n+1}$ . Denote

$$(2.21) \quad \text{S.P.}[f](\zeta) = \sum_{k=0}^m c_k(\zeta) |\zeta|^k + \sum_{k=1}^{n+1} c_{-k}(\zeta) \frac{1}{|\zeta|^k}$$

and

$$(2.22) \quad \text{I.P.}[f](\zeta) = \sum_{k=n+2}^{\infty} c_{-k}(\zeta) \frac{1}{|\zeta|^k}$$

for sufficiently large  $|\zeta|$ . If  $\text{I.P.}[f]$  is  $L^p$  integrable in the complement of a sufficiently large ball centered at the origin in  $\mathbb{R}^{n+1}$  for  $p \geq 1$ , then  $\text{S.P.}[f]$  is called the singular part of  $f$  and  $\text{I.P.}[f]$  is called the integrable part of  $f$  at infinity in the  $L^p$  sense,  $p \geq 1$ .

We immediately have

**Proposition 2.4.** Let  $f$  be defined as in Definition 2.3, then for sufficiently large  $|\zeta|$ ,

$$(2.23) \quad f(\zeta) = \text{S.P.}[f](\zeta) + \text{I.P.}[f](\zeta).$$

**Definition 2.5.** Let

$$(2.24) \quad K_m^{(j)}(x, v) = \begin{cases} D_m^{(j)}(x, v), & \text{for } |x| = |v|, \\ D_m^{(j)}(x, v) - \text{S.P.}[D_m^{(j)}](x, v), & \text{for } |x| \neq |v|, \end{cases}$$

where

$$(2.25) \quad \text{S.P.}[D_m^{(j)}](x, v) = \frac{c_n}{\beta_1 \beta_2 \cdots \beta_{m-1}} (x_j - v_j) \left[ \sum_{l=0}^{2m-1} P_l^{(\frac{n+3}{2}-m)}(x_{S^n} \cdot v_{S^n}) \right. \\ \left. \times \min \left( \left| \frac{x}{v} \right|^l, \left| \frac{x}{v} \right|^{-l} \right) \times \max (|x|^{2m-n-3}, |v|^{2m-n-3}) \right]$$

for any  $m$  and even  $n$ , or any odd  $n$  with  $m \leq \frac{n+1}{2}$ ; and

$$(2.26) \quad \text{S.P.}[D_m^{(j)}](x, v) = \frac{c_n}{(n+1)\beta_1 \beta_2 \cdots \beta_{\frac{n+1}{2}-1} \alpha_2 \alpha_4 \cdots \alpha_{2m-1}} (x_j - v_j) \\ \times \left\{ \frac{1}{2} \left[ \sum_{l=0}^{2m-1} Q_l^{(\frac{n+3}{2}-m)}(x_{S^n} \cdot v_{S^n}) \right. \right. \\ \times \min \left( \left| \frac{x}{v} \right|^l, \left| \frac{x}{v} \right|^{-l} \right) \times \max (|x|^{2m-n-3}, |v|^{2m-n-3}) \Big] \\ + \left[ \log(\max(|x|, |v|)) - \sum_{t=1}^{m-\frac{n+3}{2}} \left( \frac{1}{2t} + \frac{1}{2t+n+1} \right) \right] \\ \times \left[ \sum_{l=0}^{2m-1} P_l^{(\frac{n+3}{2}-m)}(x_{S^n} \cdot v_{S^n}) \times \min \left( \left| \frac{x}{v} \right|^l, \left| \frac{x}{v} \right|^{-l} \right) \right. \\ \left. \left. \times \max (|x|^{2m-n-3}, |v|^{2m-n-3}) \right] \right\}$$

for any odd  $n$  with  $m \geq \frac{n+3}{2}$ , in which  $\alpha_s$ ,  $\beta_s$  and  $c_n$  are given as in Lemma 2.2, and the ultraspherical polynomials  $P_l^{(\frac{n+3}{2}-m)}$ ,  $Q_l^{(\frac{n+3}{2}-m)}$  are defined by (2.16) and (2.17). Then  $K_m^{(j)}(x, v)$ ,  $1 \leq j \leq n+1$ , are said to be the  $m$ th order conjugate Poisson and Poisson kernels.

By the above definition, we immediately obtain that

**Proposition 2.6.**

$$(2.27) \quad K_m^{(j)}(x, v) = -K_m^{(j)}(v, x)$$

with  $x \neq v$  for any  $m \in \mathbb{N}$  and  $1 \leq j \leq n+1$ .

*Remark 2.7.* Let  $x = (x_1, x_2, \dots, x_n, y) \in \mathbb{R}_+^{n+1}$  and  $v = (\underline{v}, 0)$  with  $\underline{v} = (v_1, v_2, \dots, v_n)$ , then  $2K_m^{(n+1)}(x, v)$  are just the higher order Poisson kernels with a different singular part,  $G_m(x, \underline{v})$ , introduced in [22]. Using those kernels, we have resolved the following polyharmonic Dirichlet problems with  $L^p$  data in the upper-half space,  $\mathbb{R}_+^{n+1}$

$$(2.28) \quad \begin{cases} \Delta^m u = 0 \text{ in } \mathbb{R}_+^{n+1} \\ \Delta^j u = f_j \text{ on } \partial \mathbb{R}_+^{n+1} = \mathbb{R}^n, \end{cases}$$

where  $n \geq 2$ ,  $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times \mathbb{R}_+ = \{x = (\underline{x}, y) : \underline{x} \in \mathbb{R}^n, y \in \mathbb{R}, y > 0\}$ ,  $\underline{x} = (x_1, \dots, x_n)$ ,  $\Delta \equiv \Delta_{n+1} := \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y^2}$ ,  $f_j \in L^p(\mathbb{R}^n)$ ,  $m \in \mathbb{N}$ ,  $0 \leq j < m$ , and  $p \geq 1$ .

3. MULTI-LAYER  $\mathcal{D}$ -POTENTIALS

With the aforementioned preliminaries, in the present section, we introduce one class of multi-layer potentials in terms of the higher order conjugate Poisson and Poisson kernels, which are higher order analogues of the classical double layer potential.

Let  $X = (x_1, x_2, \dots, x_{n+1})$ ,  $Y = (y_1, y_2, \dots, y_{n+1}) \in \mathbb{R}^{n+1}$  and  $X \neq Y$ , for any natural number  $m \geq 1$ , define

$$(3.1) \quad K_m(X, Y) = (K_m^{(1)}(X, Y), K_m^{(2)}(X, Y), \dots, K_m^{(n+1)}(X, Y)),$$

where  $K_m^{(j)}$ ,  $1 \leq j \leq n+1$ , are the  $m$ th order conjugate Poisson and Poisson kernels.  $K_m$  is called the  $m$ th order Poisson field.

**Definition 3.1.** Let  $D$  be a simply connected (bounded or unbounded) domain in  $\mathbb{R}^{n+1}$  with the boundary  $\partial D$  and  $k \in \mathbb{N} \cup \{\infty\}$ ,  $C^k(D)$  denotes the set of the functions that have continuous partial derivatives of order  $k$  in  $D$ . If  $f$  is a continuous function defined on  $D \times \partial D$  satisfying  $f(\cdot, v) \in C^k(D)$  for any fixed  $v \in \partial D$  and  $f(x, \cdot) \in C(\partial D)$  for any fixed  $x \in D$ , then  $f$  is said to be  $C^k \times C$  on  $D \times \partial D$  and written as  $f \in (C^k \times C)(D \times \partial D)$ . When  $f$  is vector-valued,  $f \in (C^k \times C)(D \times \partial D)$  means that all of its components are in  $(C^k \times C)(D \times \partial D)$ .

**Definition 3.2.** Let  $D$  be a Lipschitz domain in  $\mathbb{R}^{n+1}$ , with the boundary  $\partial D$ . Set

$$(3.2) \quad M_j f(X) = \int_{\partial D} \langle K_j(X, Q), n_Q \rangle f(Q) d\sigma(Q), \quad X \in D,$$

where  $1 \leq j < \infty$ ,  $K_j$  is the  $j$ th order Poisson field,  $n_Q$  is the outward unit normal at  $Q \in \partial D$ ,  $\langle \cdot, \cdot \rangle$  is the inner product in  $\ell^2(\mathbb{R}^{n+1})$ ,  $d\sigma$  is the surface measure on  $\partial D$ , and  $f \in L^p(\partial D)$  for some suitable  $p$ .  $M_j f$  is called the  $j$ th-layer  $\mathcal{D}$ -potential of  $f$ .

*Remark 3.3.* By the above definition,  $M_1 f$  is the classical double layer potential.

Define

$$(3.3) \quad Tf(P) = \lim_{\epsilon \rightarrow 0} \int_{\partial D \setminus B_\epsilon(P)} \langle K_1(P, Q), n_Q \rangle f(Q) d\sigma(Q), \quad P \in \partial D,$$

where  $B_\epsilon(P) = \{Q \in \mathbb{R}^{n+1} : |Q - P| < \epsilon\}$ . Hence the adjoint operator of  $T$  is given by

$$(3.4) \quad T^*f(P) = \lim_{\epsilon \rightarrow 0} \int_{\partial D \setminus B_\epsilon(P)} \langle K_1(Q, P), n_P \rangle f(Q) d\sigma(Q), \quad P \in \partial D.$$

Due to Dahlberg, Kenig and Verchota et al., we have

**Lemma 3.4** ([12, 53]). *There exists  $\varepsilon = \varepsilon(D) > 0$  such that  $\pm \frac{1}{2}I - T$  is invertible in  $L^p(\partial D)$ ,  $2 - \varepsilon < p < \infty$ , and  $\pm \frac{1}{2}I - T^*$  is invertible in  $L^p(\partial D)$ ,  $1 < p < 2 + \varepsilon$ .*

By the properties of higher order conjugate Poisson and Poisson kernels, we have

**Theorem 3.5.** *Let  $\{K_m\}_{m=1}^\infty$  be the sequence of the Poisson fields, and  $D$  be a Lipschitz graphic domain in  $\mathbb{R}^{n+1}$ , i.e.,*

$$(3.5) \quad D = \{(\underline{x}, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > \varphi(\underline{x}), \underline{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n\},$$

where  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  is Lipschitz continuous; namely,  $|\varphi(\underline{x}) - \varphi(\underline{x}')| \leq L|\underline{x} - \underline{x}'|$  with  $0 < L < \infty$ , and set  $\varphi(0) > 0$ , then



- (1) For all  $m \in \mathbb{N}$ ,  $K_m \in (C^\infty \times C)(D \times \partial D)$ , the non-tangential boundary value

$$\lim_{\substack{X \rightarrow P \\ X \in \Gamma(P), Q \in \partial D}} K_m(X, Q) = K_m(P, Q)$$

exists for all  $P \in \partial D$  and  $P \neq Q \in \partial D$ ;  $K_m(\cdot, P)$  can be continuously extended to  $\overline{D} \setminus \{P\}$  for any fixed  $P \in \partial D$ ;

- (2) For  $m \geq 2$ ,

$$|K_m(X, Q)| \leq M \frac{|X - Q|}{(1 + |Q|^2)^{\frac{n+2+\epsilon}{2}}}$$

for any  $(X, Q) \in D_c \times \{Q \in \partial D : |Q| > T\}$ , where  $0 < \epsilon < 1$ ,  $D_c$  is any compact subset of  $\overline{D}$ ,  $T$  is a sufficiently large positive real number and  $M$  denotes some positive constant depending only on  $\epsilon, D_c$  and  $T$ ;

- (3)  $\Delta_X K_1(X, Y) = -\Delta_Y K_1(X, Y) = 0$  and  $\Delta_X K_m(X, Y) = -\Delta_Y K_m(X, Y) = K_{m-1}(X, Y)$  for any  $m > 1$ ,  $X, Y \in \mathbb{R}^{n+1} \setminus \{0\}$  and  $X \neq Y$ , where  $\Delta_X = \sum_{j=1}^{n+1} \frac{\partial}{\partial x_j^2}$  and  $\Delta_Y = \sum_{j=1}^{n+1} \frac{\partial}{\partial y_j^2}$ ;

- (4) The non-tangential limit

$$(3.6) \quad \lim_{\substack{X \rightarrow P \\ X \in \Gamma(P)}} \int_{\partial D} \langle K_1(X, Q), n_Q \rangle f(Q) d\sigma(Q) = \frac{1}{2} f(P) + T f(P),$$

for any  $f \in L^p(\partial D)$ ,  $1 \leq p < \infty$ ;

- (5) The non-tangential limit

$$(3.7) \quad \lim_{\substack{X \rightarrow P \\ X \in \Gamma(P)}} \int_{\partial D} \langle K_m(X, Q), n_Q \rangle f(Q) d\sigma(Q) = K_m f(P)$$

for any  $m \geq 2$  and  $f \in L^p(\partial D)$ ,  $1 \leq p \leq \infty$ , where

$$(3.8) \quad K_m f(P) = \int_{\partial D} \langle K_m(P, Q), n_Q \rangle f(Q) d\sigma(Q), \quad P \in \partial D$$

which is a principle value integral defined as (3.3).

*Remark 3.6.* In this theorem and what follows, with respect to the Lipschitz graphic domains, we emphasize that the Lipschitz function  $\varphi$  should satisfy the condition  $\varphi(0) > 0$  in order to avoid  $0 \in \overline{D}$ . This is only a technical requirement to guarantee the  $L^p$ -integrability on  $\partial D$  and continuity on  $\overline{D}$  of the kernels  $K_m^{(j)}$ . If  $0 \in \overline{D}$ , we can take any fixed point  $x_0 \in \mathbb{R}^{n+1} \setminus \overline{D}$  and use it to redefine the singular parts of  $K_m^{(j)}$  in (2.25) and (2.26) with the terms  $|x|$  and  $|v|$  replaced respectively by  $|x - x_0|$  and  $|v - x_0|$ . So we do, then the above theorem and main results in the paper still hold with  $x_0$  in place of 0.

*Proof.* By using the definition of the singular part, S.P.[.], and performing similar calculations as to get (2.18) and (2.19), we get (2.25) and (2.26). Note the explicit expressions (2.25) and (2.26), it immediately follows that for any  $m \in \mathbb{N}$ ,  $K_m \in (C^\infty \times C)(D \times \partial D)$ , the non-tangential boundary value

$$\lim_{\substack{X \rightarrow P \\ X \in \Gamma(P), Q \in \partial D}} K_m(X, Q) = K_m(P, Q)$$

exists for all  $P \in \partial D$  and  $P \neq Q \in \partial D$ . Furthermore,  $K_m(\cdot, P)$  can be continuously extended to  $\overline{D} \setminus \{P\}$  for any fixed  $P \in \partial D$ , i.e., the claim (1) holds.

Note that

$$D_1^{(j)}(x, v) = -\frac{1}{\omega_n} \mathcal{P}_j(x - v) = -\frac{1}{\omega_n} \frac{x_j - v_j}{|x - v|^{n+1}}.$$

So by the definition of the singular part,

$$(3.9) \quad \text{S.P.}[D_1^{(j)}](x, v) \equiv 0.$$

Therefore

$$(3.10) \quad \langle K_1(X, Q), n_Q \rangle = \frac{1}{\omega_n} \frac{\langle Q - X, n_Q \rangle}{|X - Q|^{n+1}}.$$

Then by the theory of classical layer potentials [24, 53],

$$\lim_{\substack{X \rightarrow P \\ X \in \Gamma(P)}} \int_{\partial D} \langle K_1(X, Q), n_Q \rangle f(Q) d\sigma(Q) = \frac{1}{2} f(P) + T f(P),$$

for any  $f \in L^p(\partial D)$ ,  $1 \leq p < \infty$ . Moreover, by the definition and Taylor's expansion, for sufficiently large  $|v| > |x|$ ,

$$(3.11)$$

$$\text{I.P.}[D_m^{(j)}](x, v) = \begin{cases} A_{m,n}(x_j - v_j) C_{m,n}(x, v) \frac{1}{|v|^{n+3}}, & n \text{ even and any } m, \text{ or } n \text{ odd and } m \leq \frac{n+1}{2}, \\ B_{m,n}(x_j - v_j) \left[ \tilde{C}_{m,n}(x, v) + \hat{C}_{m,n}(x, v) \log |v| \right] \frac{1}{|v|^{n+3}}, & n \text{ odd and } m \geq \frac{n+3}{2}, \end{cases}$$

where  $A_{m,n}$  and  $B_{m,n}$  are positive constants depending only on  $m$  and  $n$ ,

$$(3.12) \quad C_{m,n}(x, v) = |x|^{2m} \left\{ \frac{d^{2m}}{dr^{2m}} \left[ (1 - 2r(x_{S^n} \cdot v_{S^n}) + r^2)^{m - \frac{n+3}{2}} \right] \right\}_{r=\theta}$$

and

$$(3.13)$$

$$\begin{aligned} \tilde{C}_{m,n}(x, v) = & |x|^{2m} \left\{ \frac{d^{2m}}{dr^{2m}} \left[ (1 - 2r(x_{S^n} \cdot v_{S^n}) + r^2)^{m - \frac{n+3}{2}} \right. \right. \\ & \left. \left. \times \left[ \frac{1}{2} \log(1 - 2r(x_{S^n} \cdot v_{S^n}) + r^2) - \sum_{t=1}^{m - \frac{n+3}{2}} \left( \frac{1}{2t} + \frac{1}{2t + n + 1} \right) \right] \right] \right\}_{r=\vartheta} \end{aligned}$$

as well as

$$(3.14) \quad \hat{C}_{m,n}(x, v) = |x|^{2m} \left\{ \frac{d^{2m}}{dr^{2m}} \left[ (1 - 2r(x_{S^n} \cdot v_{S^n}) + r^2)^{m - \frac{n+3}{2}} \right] \right\}_{r=\varrho}$$

with  $0 < \theta, \vartheta, \varrho < \frac{|x|}{|v|} < 1$ . Note that

$$(3.15) \quad \lim_{|v| \rightarrow \infty} \frac{\log |v|}{|v|^\epsilon} = 0$$

for any  $\epsilon > 0$ . Therefore, for any compact subset  $D_c$  of  $\overline{D}$  and  $X \in D_c$ , by the continuity of  $C_{m,n}$ ,  $\tilde{C}_{m,n}$  and  $\hat{C}_{m,n}$ , we have

$$(3.16) \quad |K_m^{(j)}(X, Q)| = |\text{I.P.}[D_m^{(j)}](X, Q)| \leq M \frac{|x_j - v_j|}{(1 + |Q|^2)^{\frac{n+2+\epsilon}{2}}},$$

where  $0 < \epsilon < 1$ ,  $(X, Q) \in D_c \times \{Q \in \partial D : |Q| > T\}$ ,  $T$  is a sufficiently large positive real number and  $M$  is a positive constant depending only on  $\epsilon$ ,  $D_c$  and  $T$ . Thus the claims (2) and (4) are established.

From (2.25) and (2.26), we can simply denote

$$(3.17) \quad \text{S.P.}[D_m^{(j)}](x, v) = C_m(x_j - v_j) \sum_{l=0}^{2m-1} c_{m,l}(x, v) |v|^{2m-n-3-l},$$

where  $C_m$  is a constant depending only on  $m, n$ , and the coefficient functions  $c_{m,l}$  can be explicitly expressed by the ultraspherical polynomials  $P_l^{(\frac{n+3}{2}-m)}(x_{S^n} \cdot v_{S^n})$ ,  $Q_l^{(\frac{n+3}{2}-m)}(x_{S^n} \cdot v_{S^n})$ ,  $|x|^l$  and  $\log |v|$ . Therefore,

$$(3.18) \quad \Delta \left[ \text{S.P.}[D_m^{(j)}](x, v) \right] = C_m \sum_{l=0}^{2m-1} \Delta [(x_j - v_j) c_{m,l}(x, v)] |v|^{2m-n-3-l}.$$

By Lemma 2.2, we have

$$(3.19) \quad \Delta K_m^{(j)} - K_{m-1}^{(j)} = \text{S.P.}[D_{m-1}^{(j)}] - \Delta \left[ \text{S.P.}[D_m^{(j)}] \right]$$

for any  $m \geq 2$ . Due to (3.16) and (3.17), for sufficiently large  $v$  (in deed, for all  $v$ ),

$$\Delta K_m^{(j)} = K_{m-1}^{(j)} \quad \text{and} \quad \text{S.P.}[D_{m-1}^{(j)}] = \Delta \left[ \text{S.P.}[D_m^{(j)}] \right]$$

for any  $m \geq 2$ . By taking into account  $\Delta K_1 = 0$ , and by Proposition 2.6, the claim (3) follows.

Finally, we show that the claim (5) holds.

**Case 1:**  $2 \leq m \leq \frac{n+1}{2}$ . Take a splitting,

$$(3.20) \quad \begin{aligned} \int_{\partial D} \langle K_m(X, Q), n_Q \rangle f(Q) d\sigma(Q) &= \int_{\partial D \cap B_\delta(P)} \langle K_m(X, Q), n_Q \rangle f(Q) d\sigma(Q) \\ &\quad + \int_{\partial D \cap B_T(P) \setminus B_\delta(P)} \langle K_m(X, Q), n_Q \rangle f(Q) d\sigma(Q) \\ &\quad + \int_{\partial D \setminus B_T(P)} \langle K_m(X, Q), n_Q \rangle f(Q) d\sigma(Q) \\ &\triangleq \text{I} + \text{II} + \text{III}, \end{aligned}$$

where  $P$  is any fixed point in  $\partial D$ ,  $\delta, T > 0$ ,  $\delta$  is sufficiently small while  $T$  is sufficiently large,  $X \in \Gamma_{\gamma, \eta}(P) = \{X \in \Gamma_\gamma(P) : \text{dist}(X, \partial D) \leq \eta\}$ ,  $0 < \eta < \min\{\delta, \frac{1}{2}\}$ , and  $f \in L^p(\partial D)$ ,  $1 \leq p \leq \infty$ . By the claim (1),  $K_m^{(j)}(X, Q)$  is continuous on the compact set  $\Gamma_{\gamma, \eta}(P) \times \{Q \in \partial D : \delta \leq |Q - P| \leq T\}$ . Therefore,

$$(3.21) \quad \text{II} \rightarrow \int_{\partial D \cap B_T(P) \setminus B_\delta(P)} \langle K_m(P, Q), n_Q \rangle f(Q) d\sigma(Q) \quad \text{as } X \rightarrow P, \quad X \in \Gamma_{\gamma, \eta}(P).$$

By the claim (2), for sufficiently large  $T$  and some fixed  $0 < \epsilon_0 < 1$ ,  $X \in \Gamma_{\gamma, \eta}(P)$  and  $|Q - P| > T$ , we have

$$|K_m^{(j)}(X, Q)| \leq M \frac{|x_j - v_j|}{(1 + |Q|^2)^{\frac{n+2+\epsilon_0}{2}}},$$

where  $M$  is a constant depending only on  $\delta, T$  and  $\epsilon_0$ . So

$$(3.22) \quad |\langle K_m(X, Q), n_Q \rangle f(Q)| \leq M \frac{|X - Q|}{(1 + |Q|^2)^{\frac{n+2+\epsilon_0}{2}}} |f(Q)|.$$

The RHS of the above inequality belongs to  $L^1(\partial D)$ , because  $\frac{|X-Q|}{(1+|Q|^2)^{\frac{n+2+\epsilon_0}{2}}} \in L^q(\partial D) \cap C_0(\partial D)$  and  $f \in L^p(\partial D)$  for any  $1 \leq p \leq \infty$  and  $q \geq 1$ , where  $X \in \Gamma_{\gamma,\eta}(P)$  and  $C_0(\partial D)$  is the set of all continuous functions defined on  $\partial D$  vanishing at infinity. Since by (3.22),

$$\langle K_m(X, Q), n_Q \rangle f(Q) \rightarrow \langle K_m(P, Q), n_Q \rangle f(Q)$$

as  $X \rightarrow P$  for any  $X \in \Gamma_{\gamma,\eta}(P)$  and  $|Q - P| > T$ , and Lebesgue's dominated convergence theorem,

$$(3.23) \quad \text{III} \rightarrow \int_{\partial D \setminus B_T(P)} \langle K_m(P, Q), n_Q \rangle f(Q) d\sigma(Q) \text{ as } X \rightarrow P, X \in \Gamma_{\gamma,\eta}(P).$$

Write that

$$(3.24) \quad \begin{aligned} \mathbf{I}^{(j)} &= \int_{\partial D \cap B_\delta(P)} D_m^{(j)}(X, Q) n_Q^{(j)} f(Q) d\sigma(Q) \\ &\quad - \int_{\partial D \cap B_\delta(P)} \text{S.P.}[D_m^{(j)}](X, Q) n_Q^{(j)} f(Q) d\sigma(Q) \\ &\triangleq \mathbf{I}_1^{(j)} - \mathbf{I}_2^{(j)}. \end{aligned}$$

Similarly to (3.21), by taking into account  $\text{S.P.}[D_m^{(j)}](X, Q) \in C(\Gamma_{\gamma,\eta}(P) \times \{Q \in \partial D : |Q - P| \leq \delta\})$ ,

$$(3.25) \quad \mathbf{I}_2^{(j)} \rightarrow \int_{\partial D \cap B_\delta(P)} \text{S.P.}[D_m^{(j)}](P, Q) n_Q^{(j)} f(Q) d\sigma(Q) \text{ as } X \rightarrow P, X \in \Gamma_{\gamma,\eta}(P).$$

For  $X \in \Gamma_{\gamma,\eta}(P)$  and  $|Q - P| < \delta < \frac{1}{2}$ ,

$$(3.26) \quad \begin{aligned} D_m^{(j)}(X, Q) &= d_m \frac{|x_j - v_j|}{|X - Q|^{n+3-2m}} \\ &= d_m \frac{|x_j - v_j|}{\left[|Q - P|^2 + |X - P|^2 - 2(X - P) \cdot (Q - P)\right]^{\frac{n+3}{2}-m}} \\ &\leq d_m \frac{|x_j - v_j|}{\left[|Q - P|^2 + |X - P|(1 - 2|Q - P|)\right]^{\frac{n+3}{2}-m}} \\ &\leq d_m \frac{|x_j - v_j|}{|Q - P|^{(n+3)-2m}}, \end{aligned}$$

where  $d_m = \frac{c_n}{\beta_1 \beta_2 \cdots \beta_{m-1}}$ . Therefore,

$$(3.27) \quad |\mathbf{I}_1^{(j)}| \leq d_m \int_{\partial D \cap B_\delta(P)} |x_j - v_j| \frac{1}{|Q - P|^{(n+3)-2m}} |f(Q)| d\sigma(Q).$$

Since  $2 \leq (n+3) - 2m \leq n-1$  (as  $n = 2$ , we only need the second inequality), then

$$(3.28) \quad \mathbf{I}_1^{(j)} \rightarrow \int_{\partial D \cap B_\delta(P)} D_m^{(j)}(P, Q) n_Q^{(j)} f(Q) d\sigma(Q) \text{ as } X \rightarrow P, X \in \Gamma_{\gamma,\eta}(P).$$

Therefore, in this case, by (3.20), (3.21), (3.23)-(3.25), (3.28),

$$\lim_{\substack{X \rightarrow P \\ X \in \Gamma_\gamma(P)}} \int_{\partial D} \langle K_m(X, Q), n_Q \rangle f(Q) d\sigma(Q) = K_m f(P),$$

for any  $f \in L^p(\partial D)$ ,  $1 \leq p \leq \infty$ .

**Case 2:**  $m \geq \frac{n+3}{2}$ . For sufficiently large  $T > 0$ , we can split

$$(3.29) \quad \int_{\partial D} K_m^{(j)}(X, Q) n_Q^{(j)} f(Q) d\sigma(Q) = \int_{\partial D \cap B_T(P)} K_m^{(j)}(X, Q) n_Q^{(j)} f(Q) d\sigma(Q) \\ + \int_{\partial D \setminus B_T(P)} K_m^{(j)}(X, Q) n_Q^{(j)} f(Q) d\sigma(Q) \\ \triangleq J_1^{(j)} + J_2^{(j)},$$

where

$$(3.30) \quad J_1^{(j)} = \int_{\partial D \cap B_T(P)} K_m^{(j)}(X, Q) n_Q^{(j)} f(Q) d\sigma(Q)$$

$$(3.31) \quad = \int_{\partial D \cap B_T(P)} D_m^{(j)}(X, Q) n_Q^{(j)} f(Q) d\sigma(Q) \\ - \int_{\partial D \cap B_T(P)} \text{S.P.}[D_m^{(j)}](X, Q) n_Q^{(j)} f(Q) d\sigma(Q) \\ \triangleq J_{11}^{(j)} - J_{12}^{(j)}.$$

Similarly to (3.23) and (3.25), we have

$$(3.32) \quad J_2^{(j)} \rightarrow \int_{\partial D \setminus B_T(P)} K_m^{(j)}(P, Q) n_Q^{(j)} f(Q) d\sigma(Q) \text{ as } X \rightarrow P, X \in \Gamma_{\gamma, \eta}(P)$$

and

$$(3.33) \quad J_{12}^{(j)} \rightarrow \int_{\partial D \cap B_T(P)} \text{S.P.}[D_m^{(j)}](P, Q) n_Q^{(j)} f(Q) d\sigma(Q) \text{ as } X \rightarrow P, X \in \Gamma_{\gamma, \eta}(P).$$

Since  $m \geq \frac{n+3}{2}$ , by (2.10) and (2.11),  $D_m^{(j)}(X, Q) \in C(\Gamma_{\gamma, \eta}(P) \times \{Q \in \partial D : |Q - P| \leq T\})$ . Similarly to (3.28) (indeed, even more directly),

$$(3.34) \quad J_{11}^{(j)} \rightarrow \int_{\partial D \cap B_T(P)} D_m^{(j)}(P, Q) n_Q^{(j)} f(Q) d\sigma(Q) \text{ as } X \rightarrow P, X \in \Gamma_{\gamma, \eta}(P).$$

By (3.32)-(3.34), we have

$$\lim_{\substack{X \rightarrow P \\ X \in \Gamma_{\gamma}(P)}} \int_{\partial D} \langle K_m(X, Q), n_Q \rangle f(Q) d\sigma(Q) = K_m f(P),$$

for any  $f \in L^p(\partial D)$ ,  $1 \leq p \leq \infty$ .

We thus conclude the claim (5) and the proof is complete.  $\square$

**3.1.  $L^p$  bounded properties of operators  $K_m$  and multi-layer  $\mathcal{D}$ -potentials  $M_j$ .** In this section, we study the  $L^p$  bounded properties of the operators  $K_m$  given in (3.8) and the multi-layer  $\mathcal{D}$ -potentials  $M_j$  defined by (3.2), which are very significant for the solving program in this paper.

To state the main results, we first introduce some necessary notions and notations which used thoroughly in the present section and what follows.

Let  $D$  be a Lipschitz graphic domain as in Theorem 3.5 and  $w$  be a weight on  $\partial D$ , that is, a nonnegative locally integrable function on  $\partial D$  with values in  $(0, \infty)$  almost everywhere. For any  $k, \alpha \geq 0$  and, if the weight  $w$  on  $\partial D$  satisfy

$$(1) \quad [|Q|^{k+\alpha} (1 + |\log |Q||)] w^{-1}(Q) \in L^\infty(\partial D);$$

(2)  $[|Q|^k (1 + |\log |Q||)]^p |Q|^{\alpha-(p-1)n} w^{-1}(Q) \in L^{\frac{1}{p-1}}(\partial D)$  as  $p \geq 1$ ,

then  $w$  is called to be a  $(p, k, \alpha)$ -weight on  $\partial D$  and denote that  $w \in \mathcal{W}^{p,k,\alpha}(\partial D)$  (Note that the above two conditions are the same as  $p = 1$ ). Here  $\mathcal{W}^{p,k,\alpha}(\partial D)$  is the space consisting of all  $(p, k, \alpha)$ -weights on  $\partial D$ . It is easy to know that the spaces  $\mathcal{W}^{p,k,\alpha}(\partial D)$  increases as  $p, k$  and  $\alpha$  decrease. That is,

**Proposition 3.7.** *Let  $D$  be a Lipschitz graphic domain as in Theorem 3.5, then*

$$(3.35) \quad \mathcal{W}^{p,k,\alpha}(\partial D) \subset \mathcal{W}^{p,l,\alpha}(\partial D) \subset \mathcal{W}^{q,l,\alpha}(\partial D) \subset \mathcal{W}^{q,l,\beta}(\partial D)$$

when  $p > q > 1$ ,  $k > l$  and  $\alpha > \beta$ . Moreover,  $\mathcal{W}^{1,k,\alpha}(\partial D) \subset \mathcal{W}^{1,l,\alpha}(\partial D) \subset \mathcal{W}^{1,l,\beta}(\partial D)$  as  $k > l$  and  $\alpha > \beta$ .

*Proof.* Note that  $0 \notin \partial D$  and  $|Q| \geq d_0$  for any  $Q \in \partial D$ . Therefore, when  $k > l$ , we have

$$\begin{aligned} & \int_{\partial D} \left\{ [|Q|^l (1 + |\log |Q||)]^p |Q|^{\alpha} w^{-1}(Q) \right\}^{\frac{1}{p-1}} |Q|^{-n} d\sigma(Q) \\ &= \int_{\partial D} \frac{1}{|Q|^{\frac{k-l}{p-1}}} \left\{ [|Q|^k (1 + |\log |Q||)]^p |Q|^{\alpha} w^{-1}(Q) \right\}^{\frac{1}{p-1}} |Q|^{-n} d\sigma(Q) \\ &\leq d_0^{-\frac{k-l}{p-1}} \int_{\partial D} \left\{ [|Q|^k (1 + |\log |Q||)]^p |Q|^{\alpha} w^{-1}(Q) \right\}^{\frac{1}{p-1}} |Q|^{-n} d\sigma(Q) \end{aligned}$$

in which  $p > 1$ ; and similarly as  $p = 1$ ,

$$[|Q|^{l+\alpha} (1 + |\log |Q||)] w^{-1}(Q) \leq d_0^{-(k-l)} [|Q|^{k+\alpha} (1 + |\log |Q||)] w^{-1}(Q).$$

When  $p > q > 1$ , we have

$$\begin{aligned} & \int_{\partial D} \left\{ [|Q|^l (1 + |\log |Q||)]^q |Q|^{\alpha} w^{-1}(Q) \right\}^{\frac{1}{q-1}} |Q|^{-n} d\sigma(Q) \\ &= \int_{\partial D} [|Q|^l (1 + |\log |Q||)]^{(1+\frac{1}{q-1})} (|Q|^{\alpha} w^{-1}(Q))^{\frac{1}{q-1}} |Q|^{-n} d\sigma(Q) \\ &= \int_{\partial D} [|Q|^l]^{(1+\frac{1}{p-1})} (|Q|^{\alpha} w^{-1}(Q))^{\frac{1}{p-1}} \\ &\quad \times \left\{ [|Q|^{l+\alpha} (1 + |\log |Q||)] w^{-1}(Q) \right\}^{(\frac{1}{q-1} - \frac{1}{p-1})} |Q|^{-n} d\sigma(Q) \\ &\leq \|w\|_{1,l,\alpha}^{\frac{p-q}{(p-1)(q-1)}} \int_{\partial D} \left\{ [|Q|^l (1 + |\log |Q||)]^p |Q|^{\alpha} w^{-1}(Q) \right\}^{\frac{1}{p-1}} |Q|^{-n} d\sigma(Q), \end{aligned}$$

where

$$\|w\|_{1,l,\alpha} = \sup_{Q \in \partial D} \left\{ [|Q|^{l+\alpha} (1 + |\log |Q||)] w^{-1}(Q) \right\}.$$

When  $\alpha > \beta$  and  $q > 1$ , we have

$$\begin{aligned} & \int_{\partial D} \left\{ [|Q|^l (1 + |\log |Q||)]^q |Q|^{\beta} w^{-1}(Q) \right\}^{\frac{1}{q-1}} |Q|^{-n} d\sigma(Q) \\ &= \int_{\partial D} |Q|^{\frac{\beta-\alpha}{q-1}} \left\{ [|Q|^l (1 + |\log |Q||)]^q |Q|^{\alpha} w^{-1}(Q) \right\}^{\frac{1}{q-1}} |Q|^{-n} d\sigma(Q) \\ &\leq d_0^{-\frac{\alpha-\beta}{q-1}} \int_{\partial D} \left\{ [|Q|^l (1 + |\log |Q||)]^p |Q|^{\alpha} w^{-1}(Q) \right\}^{\frac{1}{q-1}} |Q|^{-n} d\sigma(Q); \end{aligned}$$

and similarly as  $q = 1$ ,

$$[|Q|^{l+\beta} (1 + |\log |Q||)] w^{-1}(Q) \leq d_0^{-(\alpha-\beta)} [|Q|^{l+\alpha} (1 + |\log |Q||)] w^{-1}(Q).$$

Thus this proposition is completed.  $\square$

*Remark 3.8.* Moreover, by the condition (1) in the definition of  $(p, k, \alpha)$ -weights, it is easy to find that the weighted function spaces,  $L^p(\partial D, w d\sigma)$  with  $w \in \mathcal{W}^{p,k,\alpha}(\partial D)$ , are some subspaces of  $L^p(\partial D)$  for any  $1 \leq p < \infty$  and  $k, \alpha \geq 0$ .

Before stating the main results, we establish the following elementary and useful lemma.

**Lemma 3.9.** *Assume that  $p \geq 1$  and  $R \geq c_0$  with positive constant  $c_0$  fixed, then*

$$(3.36) \quad \int_0^R r |\log r|^p dr \leq CR^2 [1 + |\log R|^p]$$

and

$$(3.37) \quad \int_R^\infty \frac{|\log r|^p}{r^2} dr \leq C' \frac{1}{\sqrt{R}} (1 + |\log R|^p),$$

where the constants  $C$  and  $C'$  depend only on  $p$  and  $c_0$ .

*Proof.* At first, we estimate (3.36). If  $0 < R < 1$ , then

$$(3.38) \quad \begin{aligned} \int_0^R r |\log r|^p dr &= \int_0^R |r^{\frac{1}{p}} \log r|^p dr = \int_0^R \left(-r^{\frac{1}{p}} \log r\right)^p dr \\ &\leq \left(\frac{p}{e}\right)^p R \leq c_0^{-1} \left(\frac{p}{e}\right)^p R^2 \\ &\leq c_0^{-1} \left(\frac{p}{e}\right)^p R^2 [1 + |\log R|^p]; \end{aligned}$$

while  $R \geq 1$ , then

$$(3.39) \quad \begin{aligned} \int_0^R r |\log r|^p dr &= \int_0^1 \left(-r^{\frac{1}{p}} \log r\right)^p dr + \int_1^R r (\log r)^p dr \\ &\leq \int_0^R \left[\left(\frac{p}{e}\right)^p + R (\log R)^p\right] dr; \\ &= \left(\frac{p}{e}\right)^p R + R^2 (\log R)^p \\ &\leq C_p R^2 [1 + |\log R|^p] \end{aligned}$$

where  $C_p = \max\left\{\left(\frac{p}{e}\right)^p, 1\right\}$ . So (3.36) follows.

Next turn to (3.37). If  $R \geq 1$ , we have that

$$(3.40) \quad \begin{aligned} \int_R^\infty \frac{|\log r|^p}{r^2} dr &= \int_R^\infty \frac{(\log r)^p}{r^2} dr = \int_R^\infty \frac{(r^{-\frac{1}{2p}} \log r)^p}{r^{\frac{3}{2}}} dr \\ &\leq 2^{p+1} \left(\frac{p}{e}\right)^p \frac{1}{\sqrt{R}}. \end{aligned}$$

If  $0 < R < 1$ , then we have

$$\begin{aligned}
 (3.41) \quad \int_R^\infty \frac{|\log r|^p}{r^2} dr &= \int_1^\infty \frac{(\log r)^p}{r^2} dr + \int_R^1 \frac{(-\log r)^p}{r^2} dr \\
 &\leq 2^{p+1} \left(\frac{p}{e}\right)^p + |\log R|^p \left(\frac{1}{R} - 1\right) \\
 &\leq 2^{p+1} \left(\frac{p}{e}\right)^p + \frac{2}{R} |\log R|^p \\
 &\leq C'_p \frac{1}{\sqrt{R}} (1 + |\log R|^p)
 \end{aligned}$$

where  $C'_p = c_0^{-\frac{1}{2}} \max \{2^{p+1} \left(\frac{p}{e}\right)^p, 2\}$ . Therefore, (3.37) follows from the last inequalities.  $\square$

*Remark 3.10.* By observing the above proof, in fact, we get that for any  $0 < \epsilon < 1$ ,

$$(3.42) \quad \int_R^\infty \frac{|\log r|^p}{r^2} dr \leq \hat{C} \frac{1}{R^{1-\epsilon}} (1 + |\log R|^p),$$

where the constant  $\hat{C}$  depends only on  $p, c_0$  and  $\epsilon$  and satisfies that  $\lim_{\epsilon \rightarrow 0+} \hat{C} = +\infty$  and  $\lim_{\epsilon \rightarrow 1-} \hat{C} = +\infty$ .

The main object of this section is to justify

$$(3.43) \quad K_m : L^p(\partial D, w d\sigma) \longrightarrow L^p(\partial D)$$

and

$$(3.44) \quad M_j : L^p(\partial D, w' d\sigma) \longrightarrow L^p(D)$$

are bounded with

$$\|K_m f\|_{L^p(\partial D)} \leq C \|f\|_{L^p(\partial D, w d\sigma)}$$

and

$$\|M_j f\|_{L^p(D)} \leq \tilde{C} \|f\|_{L^p(\partial D, w d\sigma)},$$

where  $M_j$  is the  $j$ th-layer  $\mathcal{D}$ -potential,  $w, w'$  are appropriate  $(p, k, \alpha)$ -weights, and  $C, \tilde{C}$  are some constants depending only on  $m, n, p$  and  $D$ . More precisely, we have

**Theorem 3.11.** *Let the Lipschitz graphic domain  $D$  and the operators  $K_m$ ,  $m \geq 2$ , be the same as in Theorem 3.5,  $w \in \mathcal{W}^{p, 2m-2, \frac{1}{2}}(\partial D)$ ,  $1 \leq p < \infty$ , then*

$$(3.45) \quad \|K_m f\|_{L^p(\partial D)} \leq C \|f\|_{L^p(\partial D, w d\sigma)}$$

for any  $f \in L^p(\partial D, w d\sigma)$ , where  $C$  is a constant depending only on  $m, n, p$  and  $d_0 = \text{dist}(0, \partial D)$ . That is,  $K_m$ ,  $m \geq 2$ , are bounded from  $L^p(\partial D, w d\sigma)$  to  $L^p(\partial D)$  for any  $w \in \mathcal{W}^{p, 2m-2, \frac{1}{2}}(\partial D)$  with  $1 \leq p < \infty$ .

*Proof.* By the definition of Lipschitz domain, we can identify the space  $L^p(\partial D)$  with the weighted space  $L^p(\mathbb{R}^n, \sqrt{1 + |\nabla \varphi|^2} dx)$ . It is easy to verify that the space is comparable the standard space  $L^p(\mathbb{R}^n)$  in terms of the fact

$$(3.46) \quad \|f\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n, \sqrt{1 + |\nabla \varphi|^2} dx)} \leq \sqrt{1 + L^2} \|f\|_{L^p(\mathbb{R}^n)},$$



where  $L$  is the Lipschitz constant of  $D$ . So here we can simply regard  $L^p(\mathbb{R}^n)$  as  $L^p(\mathbb{R}^n, \sqrt{1+|\nabla\varphi|^2}dx)$  identically. Similarly, we can also identify  $L^p(\partial D, w d\sigma)$  with  $L^p(\mathbb{R}^n, w dx)$ .

For simplicity, we will use the spaces  $L^p(\mathbb{R}^n)$  and  $L^p(\mathbb{R}^n, w dx)$  to replace the spaces  $L^p(\partial D)$  and  $L^p(\partial D, w d\sigma)$  in the following argument.

By Minkowski's inequality (also for integrals) and Hölder's inequality, we have

$$\begin{aligned}
 (3.47) \quad \|K_m f\|_{L^p(\mathbb{R}^n)} &\leq C \sum_{j=1}^{n+1} \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} |K_m^{(j)}(x, y) f(y)| dy \right]^p dx \right\}^{\frac{1}{p}} \\
 &\leq C \sum_{j=1}^{n+1} \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} |K_m^{(j)}(x, y)|^p dx \right]^{\frac{1}{p}} |f(y)| dy \\
 &\leq C \sum_{j=1}^{n+1} \int_{\mathbb{R}^n} [|y|^{2m-2} (1 + |\log |y||)] |y|^{\frac{1}{p}[\frac{1}{2}-(p-1)n]} |f(y)| dy \\
 &\leq C \sum_{j=1}^{n+1} \left\{ \int_{\mathbb{R}^n} [|y|^{2m-2} (1 + |\log |y||)]^{\frac{p}{p-1}} \left( |y|^{\frac{1}{2}} w^{-1}(y) \right)^{\frac{1}{p-1}} |y|^{-n} dy \right\}^{\frac{p-1}{p}} \\
 &\quad \times \|f\|_{L^p(\mathbb{R}^n, w dx)} \\
 &= C \sum_{j=1}^{n+1} \left\{ \int_{\mathbb{R}^n} \left[ [|y|^{2m-2} (1 + |\log |y||)]^p |y|^{\frac{1}{2}} w^{-1}(y) \right]^{\frac{1}{p-1}} |y|^{-n} dy \right\}^{\frac{p-1}{p}} \\
 &\quad \times \|f\|_{L^p(\mathbb{R}^n, w dx)} \\
 &\leq C \|f\|_{L^p(\mathbb{R}^n, w dx)},
 \end{aligned}$$

since  $w \in \mathcal{W}^{p, 2m-2, \frac{1}{2}}(\partial D)$ , where the constant  $C$  depends only on  $m, n, p, d_0$ , and the following fact

$$(3.48) \quad \int_{\mathbb{R}^n} |K_m^{(j)}(x, y)|^p dx \leq C(m, n, d_0) [|y|^{2m-2} (1 + |\log |y||)]^p |y|^{\frac{1}{2}-(p-1)n}$$

has been used in the third inequality, in which the constant  $C(m, n, d_0)$  depends only on  $m, n$  and  $d_0$ .

To get (3.48), we split

$$\begin{aligned}
 (3.49) \quad \int_{\mathbb{R}^n} |K_m^{(j)}(x, y)|^p dx &= \int_{|x| \leq 2|y|} |K_m^{(j)}(x, y)|^p dx + \int_{|x| > 2|y|} |K_m^{(j)}(x, y)|^p dx \\
 &\triangleq \mathcal{I}_1(y) + \mathcal{I}_2(y).
 \end{aligned}$$

Note that

$$\begin{aligned}
 (3.50) \quad \mathcal{I}_1(y) &\leq C_p \left\{ \int_{|x| \leq 2|y|} |D_m^{(j)}(x, y)|^p dx + \left( \int_{|x| < |y|} + \int_{|y| < |x| < 2|y|} \right) |\text{S.P.}[D_m^{(j)}](x, y)|^p dx \right\} \\
 &\triangleq \mathcal{I}_{1,1}(y) + \mathcal{I}_{1,2}(y) + \mathcal{I}_{1,3}(y).
 \end{aligned}$$

Firstly, to estimate  $\mathcal{I}_{1,1}$ , we note that  $|x - y| \leq 3|y|$  when  $|x| \leq 2|y|$ , and

$$(3.51) \quad |D_m^{(j)}(x, y)| \leq C_{m,n} |x - y|^{2m-(n+2)} (1 + |\log |x - y||),$$

where  $C_{m,n}$  is a constant depending only  $m$  and  $n$ , then by (3.36),

$$\begin{aligned}
(3.52) \quad \int_{|x| \leq 2|y|} |D_m^{(j)}(x, y)|^p dx &\leq \int_{|x-y| \leq 3|y|} |D_m^{(j)}(x, y)|^p dx \\
&\leq C(m, n) \int_{|x-y| \leq 3|y|} \left[ |x-y|^{2m-(n+2)} (1 + |\log |x-y||) \right]^p dx \\
&\leq C(m, n, p) |y|^{(2m-(n+2))p} \int_{|x-y| \leq 3|y|} \left[ (1 + |\log |x-y||) \right]^p dx \\
&\leq C(m, n, p) |y|^{(2m-2)p-(p-1)n} (1 + |\log |y||)^p \\
&\leq C(m, n, p, d_0) \left[ |y|^{(2m-2)} (1 + |\log |y||) \right]^p |y|^{\frac{1}{2}-(p-1)n},
\end{aligned}$$

where  $C(\cdots)$  denotes a constant depending only on the parameters in the parenthesis, and the fact  $|y| \geq d_0$  have been used in the last inequality.

Next to estimate  $\mathcal{I}_{1,2}$ . When  $|x| < |y|$ , by the definition

$$(3.53) \quad |\text{S.P.}[D_m^{(j)}](x, y)| \leq C_{m,n} |y|^{2m-n-2} (1 + |\log |y||),$$

we have

$$\begin{aligned}
(3.54) \quad \int_{|x| < |y|} |\text{S.P.}[D_m^{(j)}](x, y)|^p dx &\leq C(m, n) [|y|^{2m-n-2} (1 + |\log |y||)]^p \text{Vol}(B(0, |y|)) \\
&\leq C(m, n) [|y|^{2m-2} (1 + |\log |y||)]^p |y|^{-(p-1)n} \\
&\leq C(m, n, d_0) [|y|^{2m-2} (1 + |\log |y||)]^p |y|^{\frac{1}{2}-(p-1)n},
\end{aligned}$$

where the fact  $|y| \geq d_0$  have been used in the last inequality.

The third to estimate  $\mathcal{I}_{1,3}$ . In this case, by the definition, as  $|y| < |x| < 2|y|$ ,

$$(3.55) \quad |\text{S.P.}[D_m^{(j)}](x, y)| \leq C_{m,n} |y|^{2m-n-2} (1 + |\log |x||),$$

then by (3.36), we obtain

$$\begin{aligned}
(3.56) \quad \int_{|y| < |x| < 2|y|} |\text{S.P.}[D_m^{(j)}](x, y)|^p dx &\leq C(m, n) |y|^{(2m-n-2)p} \int_{|y| < |x| < 2|y|} (1 + |\log |x||)^p dx \\
&\leq C(m, n, p) [|y|^{2m-2} (1 + |\log |y||)]^p |y|^{-(p-1)n} \\
&\leq C(m, n, p, d_0) [|y|^{2m-2} (1 + |\log |y||)]^p |y|^{\frac{1}{2}-(p-1)n},
\end{aligned}$$

where the fact  $|y| \geq d_0$  have been used in the last inequality.

Finally, we turn to estimate  $\mathcal{I}_2$ . Note that  $r = \frac{|y|}{|x|} \in (0, \frac{1}{2})$  as  $|x| > 2|y|$ , and  $1 - 2r(x_{S^n} \cdot y_{S^n}) + r^2 \in (\frac{1}{4}, \frac{9}{4})$  as  $r \in (0, \frac{1}{2})$ . Thus by (3.11)-(3.14) and the definition, we have

$$(3.57) \quad |\text{I.P.}[D_m^{(j)}](x, y)| \leq C_{m,n} |y|^{2m} (1 + |\log |x||) \frac{1}{|x|^{n+2}}.$$

Therefore, by (3.37), we get

(3.58)

$$\begin{aligned}
\int_{|x|>2|y|} |K_m^{(j)}(x, y)| dx &= \int_{|x|>2|y|} |\text{I.P.}[D_m^{(j)}](x, y)| dx \\
&\leq C(m, n) |y|^{2mp} \int_{|x|>2|y|} \left[ \frac{1 + |\log |x||}{|x|^{n+2}} \right]^p dx \\
&\leq C(m, n, d_0) |y|^{2mp-(n+2)(p-1)} \int_{|x|>2|y|} \frac{(1 + |\log |x||)^p}{|x|^{n+2}} dx \\
&= C(m, n, d_0) |y|^{(2m-1)p-(n+1)(p-1)} \int_{|x|>2|y|} \frac{(1 + |\log |x||)^p}{|x|^{n+1}} dx \\
&\leq C(m, n, p, d_0) |y|^{(2m-2)p-(p-1)n+1} \left( \frac{1}{|y|} + \frac{|\log |y||^p}{\sqrt{|y|}} \right) \\
&\leq C(m, n, p, d_0) [|y|^{2m-2} (1 + |\log |y||)]^p |y|^{-(p-1)n+\frac{1}{2}}
\end{aligned}$$

where the fact  $|y| \geq d_0$  have been used in the last inequality.

Therefore, (3.48) follows from (3.49), (3.50), (3.52), (3.54), (3.56), (3.58). Thus the theorem is completed.  $\square$

**Theorem 3.12.** *Let the graphic Lipschitz domain  $D$  and operators  $M_j$ ,  $j \geq 2$ , be the same as in Theorem 3.5,  $w \in \mathcal{W}^{p, 2j-2, \frac{3}{2}}(\partial D)$ ,  $1 \leq p < \infty$ , then*

$$(3.59) \quad \|M_j f\|_{L^p(D)} \leq C \|f\|_{L^p(\partial D, w d\sigma)}$$

for any  $f \in L^p(\partial D, w d\sigma)$ , where  $C$  is a constant depending only on  $m, n, p$  and  $d_0$ . That is,  $M_j$ ,  $j \geq 2$ , are bounded from  $L^p(\partial D, w d\sigma)$  to  $L^p(D)$  for any  $w \in \mathcal{W}^{p, 2j-2, \frac{3}{2}}(\partial D)$  with  $1 \leq p < \infty$ .

*Proof.* It is similar to Theorem 3.11 only with  $X \in D$  in place of  $P \in \partial D$ .  $\square$

#### 4. POLYHARMONIC DIRICHLET PROBLEMS IN LIPSCHITZ GRAPHIC DOMAINS

In this section, we solve the PHD problems (1.1), viz.,

$$(4.1) \quad \begin{cases} \Delta^m u = 0, & \text{in } D, \\ \Delta^j u = f_j, & \text{on } \partial D, \end{cases}$$

where  $u - M_1 \tilde{f}_0 \in L^p(D)$  with  $\|u - M_1 \tilde{f}_0\|_{L^p(D)} \leq C \sum_{j=1}^{m-1} \|f_j\|_{L^p(\partial D, w d\sigma)}$  in which the constant  $C$  depends only on  $m, n, p$  and  $d_0$ ,  $\Delta = \sum_{k=1}^{n+1} \frac{\partial^2}{\partial x_k^2}$ ,  $D$  is a Lipschitz graphic domain stated as in Theorem 3.5,  $f_0 \in L^p(\partial D)$  and  $f_j \in L^p(\partial D, w d\sigma)$ ,  $1 \leq j \leq m-1$  for some suitable  $p > 1$ , the  $(p, 2m-2, \frac{3}{2})$ -weight  $w$  on  $\partial D$  is given as in section 3.1,  $\tilde{f}_0$  is related to all the boundary data  $f_j$ ,  $0 \leq j < m$  and  $m \in \mathbb{N}$ .

To do so, firstly, we establish

**Lemma 4.1.** *Let  $E$  be a simply connected unbounded domain in  $\mathbb{R}^{n+1}$  with smooth boundless boundary  $\partial E$ . If  $f \in (C^1 \times C)((\mathbb{R}^{n+1} \setminus \partial E) \times \partial E)$  and there exist*

$g_0, g_1 \in L^p(\partial E)$ ,  $p \geq 1$  such that

$$(4.2) \quad |f(X, Q)| \leq M_0 \frac{g_0(Q)}{(1 + |Q|^2)^{\frac{n}{2}}}$$

and

$$(4.3) \quad \left| \frac{\partial}{\partial x_j} f(X, Q) \right| \leq M_1 \frac{g_1(Q)}{(1 + |Q|^2)^{\frac{n}{2}}}$$

hold for any  $(X, Q) \in E_c \times \{Q \in \partial E : |Q| > T\}$  and  $j = 1, 2, \dots, n+1$ , where  $E_c$  is a compact subset of  $\mathbb{R}^{n+1} \setminus \partial E$ ,  $T$  is a sufficiently large positive real number and  $M_0, M_1$  are positive constants depending only on  $E_c$  and  $T$ , then

$$(4.4) \quad \frac{\partial}{\partial x_j} \left( \int_{\partial E} f(X, Q) d\sigma(Q) \right) = \int_{\partial E} \frac{\partial f}{\partial x_j}(X, Q) d\sigma(Q), \quad X \in \mathbb{R}^{n+1} \setminus \partial E$$

for any  $1 \leq j \leq n+1$ , where  $d\sigma$  is the surface measure of  $\partial E$ .

*Proof.* Fix  $X = (x_1, x_2, \dots, x_{n+1}) \in E$  and  $j \in \{1, 2, \dots, n+1\}$ , take  $X_l = X + t_l e_j$  with  $\lim_{l \rightarrow +\infty} t_l = 0$ , and  $e_j = (0, \dots, 1, \dots, 0) \in \mathbb{R}^{n+1}$  whose the  $j$ th element is 1 and other ones are zero. Denote

$$(4.5) \quad \begin{aligned} D_l(X, Q) &= \frac{f(X_l, Q) - f(X, Q)}{t_l} \\ &= \frac{\partial}{\partial x_j} f(X + \theta t_l e_j, Q), \end{aligned}$$

where  $0 < \theta < 1$ , then by (4.3),

$$(4.6) \quad |D_l(X, Q)| \leq M_1 \frac{g_1(Q)}{(1 + |Q|^2)^{\frac{n}{2}}}$$

uniformly in  $\{Q \in \partial E : |Q| > T\}$  whenever  $X_l \in \{Y : |Y - X| \leq R\} \subset \mathbb{R}^{n+1} \setminus \partial E$  for some  $R > 0$  and sufficiently large  $T > 0$ . Since  $f \in (C^1 \times C)((\mathbb{R}^{n+1} \setminus \partial E) \times \partial E)$  and

$$(4.7) \quad \lim_{l \rightarrow +\infty} D_l(X, Q) = \frac{\partial f}{\partial x_j}(X, Q), \quad Q \in \partial E,$$

by (4.2), (4.6), the continuity of  $f$  on compact set  $\{Y : |Y - X| \leq R\} \times \{Q \in \partial D : |Q| \leq T\}$ , and Lebesgue's dominated convergence theorem,

$$(4.8) \quad \begin{aligned} \lim_{l \rightarrow +\infty} \int_{\partial E} D_l(X, Q) d\sigma(Q) &= \lim_{l \rightarrow +\infty} \left[ \int_{|Q| \leq T, Q \in \partial E} D_l(X, Q) d\sigma(Q) \right. \\ &\quad \left. + \int_{|Q| > T, Q \in \partial E} D_l(X, Q) d\sigma(Q) \right] \\ &= \int_{|Q| \leq T, Q \in \partial E} \frac{\partial f}{\partial x_j}(X, Q) d\sigma(Q) \\ &\quad + \int_{|Q| > T, Q \in \partial E} \frac{\partial f}{\partial x_j}(X, Q) d\sigma(Q) \\ &= \int_{\partial E} \frac{\partial f}{\partial x_j}(X, Q) d\sigma(Q), \end{aligned}$$

i.e.,

$$\lim_{l \rightarrow +\infty} \frac{\int_{\partial E} f(X_l, Q) d\sigma(Q) - \int_{\partial E} f(X, Q) d\sigma(Q)}{t_l} = \int_{\partial E} \frac{\partial f}{\partial x_j}(X, Q) d\sigma(Q).$$

Since  $X$  and the sequence  $X_l$  are arbitrarily chosen, then

$$\frac{\partial}{\partial x_j} \left( \int_{\partial E} f(X, Q) d\sigma(Q) \right) = \int_{\partial E} \frac{\partial f}{\partial x_j}(X, Q) d\sigma(Q)$$

for any  $1 \leq j \leq n+1$  and  $X \in \mathbb{R}^{n+1} \setminus \partial E$ .  $\square$

As an immediate consequence, we have

**Corollary 4.2.** *Let  $E$  be a simply connected unbounded domain in  $\mathbb{R}^{n+1}$  with smooth boundless boundary  $\partial E$ . If  $f \in (C^2 \times C)((\mathbb{R}^{n+1} \setminus \partial E) \times \partial E)$  and there exist  $g_0, g_1, g_2 \in L^p(\partial E)$ ,  $p \geq 1$  such that*

$$(4.9) \quad |f(X, Q)| \leq M_0 \frac{g_0(Q)}{(1 + |Q|^2)^{\frac{p}{2}}},$$

$$(4.10) \quad \left| \frac{\partial}{\partial x_j} f(X, Q) \right| \leq M_1 \frac{g_1(Q)}{(1 + |Q|^2)^{\frac{p}{2}}}$$

and

$$(4.11) \quad \left| \frac{\partial^2}{\partial x_j^2} f(X, Q) \right| \leq M_2 \frac{g_2(Q)}{(1 + |Q|^2)^{\frac{p}{2}}}$$

hold for any  $(X, Q) \in E_c \times \{Q \in \partial E : |Q| > T\}$  and  $j = 1, 2, \dots, n+1$ , where  $E_c$  is any compact subset of  $\mathbb{R}^{n+1} \setminus \partial E$ ,  $T$  is a sufficiently large positive real number and  $M_0, M_1, M_2$  are positive constants depending only on  $E_c$  and  $T$ , then

$$(4.12) \quad \Delta \left( \int_{\partial E} f(X, Q) d\sigma(Q) \right) = \int_{\partial E} \Delta f(X, Q) d\sigma(Q), \quad X \in \mathbb{R}^{n+1} \setminus \partial E.$$

From the above corollary, we can obtain the following theorem concerning the differentiability of the multi-layer  $\mathcal{D}$ -potentials.

**Theorem 4.3.** *Let  $\{K_m\}_{m=1}^\infty$  be the sequence of higher order Poisson fields as in the previous section, and  $E$  be a simply connected unbounded domain in  $\mathbb{R}^{n+1}$  with smooth boundless boundary  $\partial E$ . Then for any  $m > 1$  and  $f \in L^p(\partial E)$ ,  $p \geq 1$ ,*

$$(4.13) \quad \Delta \left( \int_{\partial E} \langle K_m(X, Q), n_Q \rangle f(Q) d\sigma(Q) \right) = \int_{\partial E} \langle K_{m-1}(X, Q), n_Q \rangle f(Q) d\sigma(Q),$$

where  $X \in \mathbb{R}^{n+1} \setminus \partial E$ , namely,

$$(4.14) \quad \Delta M_m f(X) = M_{m-1} f(X), \quad X \in \mathbb{R}^{n+1} \setminus \partial E.$$

*Proof.* From the claim (1) in Theorem 3.5 (by the same argument, the claims (1)-(3) and (5) make sense for the present domains  $E$  stated here), we know that  $K_m \in (C^2 \times C)((\mathbb{R}^{n+1} \setminus \partial E) \times \partial E)$ . For any  $1 \leq j \leq n+1$ ,

(4.15)

$$\begin{aligned} K_m^{(j)}(X, Q) &= D_m^{(j)}(X, Q) - \text{S.P.}[D_m^{(j)}](X, Q) = \text{I.P.}[D_m^{(j)}](X, Q) \\ &= (x_j - v_j) \sum_{k=2m}^{\infty} [C_{m,-k}(X, Q) + \tilde{C}_{m,-k}(X, Q) \log |Q|] \frac{1}{(1 + |Q|^2)^{\frac{k}{2} - m + \frac{n+3}{2}}}, \end{aligned}$$

for any  $(X, Q) \in (\mathbb{R}^{n+1} \setminus \partial E) \times \partial E$  with  $|X| < |Q|$ , where  $C_{m,-k}$  and  $\tilde{C}_{m,-k}$  can be explicitly expressed by the ultraspherical polynomials  $P_l^{(\frac{n+3}{2}-m)}$  and  $Q_l^{(\frac{n+3}{2}-m)}$ . So by the claim (2) in Theorem 3.5, i.e., (3.16) and similar arguments to (3.16), we obtain

$$(4.16) \quad |K_m^{(j)}(X, Q)| \leq M_0 \frac{1}{(1 + |Q|^2)^{\frac{n+1+\epsilon}{2}}},$$

$$(4.17) \quad \left| \frac{\partial}{\partial x_l} K_m^{(j)}(X, Q) \right| \leq M_1 \frac{1}{(1 + |Q|^2)^{\frac{n+1+\epsilon}{2}}}$$

and

$$(4.18) \quad \left| \frac{\partial^2}{\partial x_l^2} K_m^{(j)}(X, Q) \right| \leq M_2 \frac{1}{(1 + |Q|^2)^{\frac{n+1+\epsilon}{2}}}$$

for any  $m \geq 2$ ,  $1 \leq l \leq n+1$ ,  $0 < \epsilon < 1$ , and  $(X, Q) \in E_c \times \{Q \in \partial E : |Q| > T\}$ , where  $E_c$  is any compact subset of  $\mathbb{R}^{n+1} \setminus \partial E$ ,  $T$  is a sufficiently large positive real number and  $M_0, M_1, M_2$  are positive constants depending only on  $E_c, T$  and  $\epsilon$ . Therefore, by a similar argument as Corollary 4.2 and the claim (3) in Theorem 3.5, for any  $m > 1$ ,

$$(4.19) \quad \Delta \left( \int_{\partial E} \langle K_m(X, Q), n_Q \rangle f(Q) d\sigma(Q) \right) = \int_{\partial E} \langle K_{m-1}(X, Q), n_Q \rangle f(Q) d\sigma(Q),$$

where  $X \in \mathbb{R}^{n+1} \setminus \partial E$ , i.e.,

$$\Delta M_m f(X) = M_{m-1} f(X), \quad X \in \mathbb{R}^{n+1} \setminus \partial E. \quad \square$$

*Remark 4.4.* By the same arguments, all of the above results hold when the domains  $E$  are replaced by the Lipschitz graphic domains  $D$  stated as in Theorem 3.5.

Now we can give the main result for polyharmonic Dirichlet problems in Lipschitz graph domains as follows.

**Theorem 4.5.** *Let  $\{K_m\}_{m=1}^\infty$  be the sequence of the Poisson fields, and  $D$  be a Lipschitz graphic domain in  $\mathbb{R}^{n+1}$  with Lipschitz graphic boundary  $\partial D$  as in Theorem 3.5, then for any  $m > 1$ , there exists  $\varepsilon = \varepsilon(D) > 0$  such that the PHD problem (4.1) with the data  $f_0 \in L^p(\partial D)$  and  $f_j \in L^p(\partial D, w d\sigma)$  with  $w \in W^{p, 2m-2, \frac{3}{2}}(\partial D)$ ,  $1 \leq j \leq m-1$ ,  $2 - \varepsilon < p < \infty$ , is solvable and a solution is given by*

$$(4.20) \quad \begin{aligned} u(X) &= \sum_{j=1}^m \int_{\partial D} \langle K_j(X, Q), n_Q \rangle \tilde{f}_{j-1}(Q) d\sigma(Q), \\ &= \sum_{j=1}^m M_j \tilde{f}_{j-1}(X), \quad X \in D, \end{aligned}$$

where

$$(4.21) \quad \tilde{f}_{m-1} = \left( \frac{1}{2}I + T \right)^{-1} f_{m-1}$$

and

$$(4.22) \quad \tilde{f}_l = \left( \frac{1}{2}I + T \right)^{-1} \left( f_l - \sum_{j=l+2}^m K_{j-l} \tilde{f}_{j-1} \right)$$

with  $0 \leq l \leq m-2$ , which satisfying the following estimate

$$(4.23) \quad \|u - M_1 \tilde{f}_0\|_{L^p(D)} \leq C \sum_{j=1}^{m-1} \|f_j\|_{L^p(\partial D, w d\sigma)}.$$

Under this estimate, the solution (4.20) with (4.21) and (4.22) is unique.

*Proof.* At first, we consider the existence of solution to (4.1). Formally, denote the solution of (4.1) as follows

$$(4.24) \quad u(X) = M_1 \tilde{f}_0(X) + M_2 \tilde{f}_1(X) + \cdots + M_m \tilde{f}_{m-1}(X)$$

for some functions  $\tilde{f}_j$ ,  $0 \leq j \leq m-1$  to be determined soon, where  $M_j$  is the  $j$ th-layer  $\mathcal{D}$ -potential.

Letting the polyharmonic operators  $\Delta^l$ ,  $0 \leq l \leq m$ , acting on two sides of (4.24), by Theorem 4.3, we formally have

$$\begin{cases} u(X) &= M_1 \tilde{f}_0(X) + M_2 \tilde{f}_1(X) + M_3 \tilde{f}_2(X) + \cdots + M_m \tilde{f}_{m-1}(X), \\ \Delta u(X) &= M_1 \tilde{f}_1(X) + M_2 \tilde{f}_2(X) + \cdots + M_{m-1} \tilde{f}_{m-1}(X), \\ \Delta^2 u(X) &= M_1 \tilde{f}_2(X) + \cdots + M_{m-2} \tilde{f}_{m-1}(X), \\ &\dots \\ \Delta^{m-1} u(X) &= M_1 \tilde{f}_{m-1}(X), \\ \Delta^m u(X) &= 0. \end{cases}$$

Furthermore, let  $X \in D$  converge to  $P \in \partial D$  non-tangentially, by (3.6) and (3.7), using the boundary value data of (4.1), then

$$\begin{cases} f_0(P) &= (\frac{1}{2}I + T) \tilde{f}_0(P) + K_2 \tilde{f}_1(P) + K_3 \tilde{f}_2(P) + \cdots + K_m \tilde{f}_{m-1}(P), \\ f_1(P) &= (\frac{1}{2}I + T) \tilde{f}_1(P) + K_2 \tilde{f}_2(P) + \cdots + K_{m-1} \tilde{f}_{m-1}(P), \\ f_2(P) &= (\frac{1}{2}I + T) \tilde{f}_2(P) + \cdots + K_{m-2} \tilde{f}_{m-1}(P), \\ &\dots \\ f_{m-1}(P) &= (\frac{1}{2}I + T) \tilde{f}_{m-1}(P). \end{cases}$$

By the invertible property of  $\frac{1}{2}I + T$  and  $L^p$  boundness of  $K_m$ , then we have

$$\begin{cases} \tilde{f}_0(P) &= (\frac{1}{2}I + T)^{-1} \left[ f_0(P) - K_2 \tilde{f}_1(P) - K_3 \tilde{f}_2(P) - \cdots - K_m \tilde{f}_{m-1}(P) \right], \\ \tilde{f}_1(P) &= (\frac{1}{2}I + T)^{-1} \left[ f_1(P) - K_2 \tilde{f}_2(P) - \cdots - K_{m-1} \tilde{f}_{m-1}(P) \right], \\ \tilde{f}_2(P) &= (\frac{1}{2}I + T)^{-1} \left[ f_2(P) - \cdots - K_{m-2} \tilde{f}_{m-1}(P) \right], \\ &\dots \\ \tilde{f}_{m-1}(P) &= (\frac{1}{2}I + T)^{-1} f_{m-1}(P). \end{cases}$$

Therefore, we get

$$(4.25) \quad \begin{cases} \tilde{f}_{m-1} = (\frac{1}{2}I + T)^{-1} f_{m-1}, \\ \tilde{f}_l = (\frac{1}{2}I + T)^{-1} \left[ f_l - \sum_{j=l+2}^m K_{j-l} \tilde{f}_{j-1} \right]. \end{cases}$$

where  $0 \leq l \leq m-2$ . More concisely,

$$(4.26) \quad \tilde{f}_l = \left( \frac{1}{2}I + T \right)^{-1} \left( f_l - \sum_{j=l+2}^m K_{j-l} \tilde{f}_{j-1} \right)$$

with  $0 \leq l \leq m-1$  by the convention that  $\sum_{j=l}^k s_j = 0$  as  $k < l$ .

Noting Remark 3.8, by Lemma 3.4 and Theorem 3.11, it is noteworthy that the above formal reasoning makes sense when  $f_0 \in L^p(\partial D)$  and  $f_j \in L^p(\partial D, w d\sigma)$  with  $w \in \mathcal{W}^{p, 2m-2, \frac{3}{2}}(\partial D)$ ,  $1 \leq j \leq m-1$ ,  $2-\varepsilon < p < \infty$ , where  $\varepsilon$  is the same as in Lemma 3.4. That is, a solution of (4.1) is (4.20) with (4.21) and (4.22).

Next we turn to the estimate and uniqueness of the solution. By Theorems 3.11, 3.12, and Lemma 3.4, we have

$$\begin{aligned}
 (4.27) \quad \|u - M_1 \tilde{f}_0\|_{L^p(D)} &= \left\| \sum_{j=2}^m M_j \tilde{f}_{j-1} \right\|_{L^p(D)} \\
 &\leq \sum_{j=2}^m \|M_j \tilde{f}_{j-1}\|_{L^p(D)} \\
 &\leq C \sum_{j=1}^{m-1} \|f_j\|_{L^p(\partial D, w d\sigma)}
 \end{aligned}$$

where  $w \in \mathcal{W}^{p, 2m-2, \frac{3}{2}}(\partial D)$  with  $2-\varepsilon < p < \infty$ , and the constant  $C$  depends only on  $m, n, p$  and  $d_0$ .

So by the above estimate, the uniqueness of solution follows. Thus this theorem is completed.  $\square$

## 5. POLYHARMONIC FUNDAMENTAL SOLUTIONS

By similar computations as in Section 2, it is easy to know that

$$\Delta(|x|^s) = s(s+n-1)|x|^{s-2},$$

$$\Delta(|x|^s \log |x|) = s(s+n-1)|x|^{s-2} \log |x| + (2s+n-1)|x|^{s-2}$$

and

$$\Delta(\log |x|) = (n-1)|x|^{-2}.$$

Set

$$(5.1) \quad \delta_s = s(s+n-1),$$

therefore

$$(5.2) \quad \Delta\left(\frac{1}{\delta_s}|x|^s\right) = |x|^{s-2},$$

$$(5.3) \quad \Delta\left(\frac{1}{\delta_s}|x|^s \log |x|\right) = |x|^{s-2} \log |x| + \left(\frac{1}{s} + \frac{1}{s+n-1}\right)|x|^{s-2}$$

and

$$(5.4) \quad \Delta\left(\frac{1}{n-1} \log |x|\right) = |x|^{-2}.$$

**Lemma 5.1.** *Let*

$$(5.5) \quad \mathcal{D}_1(x, v) = \mathcal{C}_n \frac{1}{|x-v|^{n-1}}$$

where

$$(5.6) \quad \mathcal{C}_n = \frac{1}{(n-1)\omega_n}.$$



For  $m \geq 2$ ,

$$(5.7) \quad \mathcal{D}_m(x, v) = \frac{\mathcal{C}_n}{\gamma_1 \gamma_2 \cdots \gamma_{m-1}} |x - v|^{2m-(n+1)}$$

if  $n$  is even, and

$$(5.8) \quad \mathcal{D}_m(x, v) = \begin{cases} \frac{\mathcal{C}_n}{\gamma_1 \gamma_2 \cdots \gamma_{m-1}} |x - v|^{2m-(n+1)}, & m \leq \frac{n-1}{2}, \\ \frac{\mathcal{C}_n}{(n-1)\gamma_1 \gamma_2 \cdots \gamma_{\frac{n-1}{2}-1} \delta_2 \delta_4 \cdots \delta_{2m-n-1}} |x - v|^{2m-(n+1)} \\ \times \left[ \log |x - v| + \frac{1}{n+1} - \sum_{t=1}^{m-\frac{n+1}{2}} \left( \frac{1}{2t} + \frac{1}{2t+n-1} \right) \right], & m \geq \frac{n+1}{2} \end{cases}$$

if  $n$  is odd, where

$$(5.9) \quad \gamma_k = \delta_{2k-n+1}, \quad k = 1, 2, \dots, m-1.$$

Then

$$(5.10) \quad \Delta \mathcal{D}_1(x, v) = 0 \text{ and } \Delta \mathcal{D}_m(x, v) = \mathcal{D}_{m-1}(x, v), \quad m \geq 2.$$

*Proof.* Using (5.2)-(5.4), it is immediate by a straightforward calculation.  $\square$

**Definition 5.2.** Let

$$(5.11) \quad \mathcal{K}_m(x, v) = \begin{cases} \mathcal{D}_m(x, v), & \text{for } |x| = |y|, \\ \mathcal{D}_m(x, v) - \text{S.P.}[\mathcal{D}_m](x, v), & \text{for } |x| \neq |y| \end{cases}$$

where

$$(5.12) \quad \text{S.P.}[\mathcal{D}_m^{(j)}](x, v) = \frac{\mathcal{C}_n}{\gamma_1 \gamma_2 \cdots \gamma_{m-1}} \left[ \sum_{l=0}^{2m} P_l^{(\frac{n+1}{2}-m)}(x_{S^n} \cdot v_{S^n}) \right. \\ \left. \times \min \left( \left| \frac{x}{v} \right|^l, \left| \frac{x}{v} \right|^{-l} \right) \times \max(|x|^{2m-n-1}, |v|^{2m-n-1}) \right]$$

for any  $m$  and even  $n$ , or any odd  $n$  with  $m \leq \frac{n-1}{2}$ ; and

(5.13)

$$\begin{aligned} \text{S.P.}[\mathcal{D}_m](x, v) &= \frac{\mathcal{C}_n}{(n-1)\gamma_1 \gamma_2 \cdots \gamma_{\frac{n-1}{2}-1} \delta_2 \delta_4 \cdots \delta_{2m-n-1}} \\ &\times \left\{ \frac{1}{2} \left[ \sum_{l=0}^{2m} Q_l^{(\frac{n+1}{2}-m)}(x_{S^n} \cdot v_{S^n}) \times \min \left( \left| \frac{x}{v} \right|^l, \left| \frac{x}{v} \right|^{-l} \right) \right. \right. \\ &\times \max(|x|^{2m-n-1}, |v|^{2m-n-1}) \Big] \\ &+ \left[ \log(\max(|x|, |v|)) + \frac{1}{n+1} - \sum_{t=1}^{m-\frac{n+1}{2}} \left( \frac{1}{2t} + \frac{1}{2t+n-1} \right) \right] \\ &\times \left[ \sum_{l=0}^{2m} P_l^{(\frac{n+1}{2}-m)}(x_{S^n} \cdot v_{S^n}) \times \min \left( \left| \frac{x}{v} \right|^l, \left| \frac{x}{v} \right|^{-l} \right) \right. \\ &\times \max(|x|^{2m-n-1}, |v|^{2m-n-1}) \Big] \Big\} \end{aligned}$$

for any odd  $n$  with  $m \geq \frac{n+1}{2}$ , where  $\delta_s, \gamma_s, \mathcal{C}_n$  are given as in (5.1) and Lemma 5.1, and the ultraspherical polynomials  $P_l^{(\frac{n+1}{2}-m)}, Q_l^{(\frac{n+1}{2}-m)}$  are defined by (2.16) and (2.17). Then  $-\mathcal{K}_m(x, v)$  is said to be the  $m$ th order polyharmonic fundamental solution.

As Proposition 2.6, by the above definition, we have

**Proposition 5.3.**

$$(5.14) \quad \mathcal{K}_m(x, v) = \mathcal{K}_m(v, x)$$

with  $x \neq v$  for any  $m \in \mathbb{N}$ .

The following theorem exhibits a nice relation between the higher order Poisson and conjugate Poisson kernels and the higher order polyharmonic fundamental solutions.

**Theorem 5.4.** *Let  $\mathcal{K}_m$  and  $K_m^{(j)}$  be as above, then*

$$(5.15) \quad \frac{\partial}{\partial x_j} \mathcal{K}_m(x, v) = K_m^{(j)}(x, v)$$

and

$$(5.16) \quad \frac{\partial}{\partial v_j} \mathcal{K}_m(x, v) = K_m^{(j)}(x, v)$$

for any  $x, v \in \mathbb{R}^{n+1} \setminus \{x \neq v\}$  and  $1 \leq j \leq n+1$ .

*Proof.* By the symmetry in Proposition 5.3, it is enough to prove (5.15). To do so, at first, we claim that

$$(5.17) \quad \frac{\partial}{\partial x_j} \mathcal{D}_m(x, v) = D_m^{(j)}(x, v)$$

for any  $x, v \in \mathbb{R}^{n+1} \setminus \{x \neq v\}$  and  $1 \leq j \leq n+1$ .

Noting (2.5) and (5.1), we have

$$(5.18) \quad \delta_s = \frac{s}{s-2} \alpha_{s-2}$$

for any odd  $s$ . To get (5.17), we consider the following three cases.

**Case I:**  $m \geq 2$  with even  $n$ , or  $m \leq \frac{n-1}{2}$  with odd  $n$ .

(5.19)

$$\begin{aligned}
\frac{\partial}{\partial x_j} \mathcal{D}_m(x, v) &= \frac{\partial}{\partial x_j} \left[ \frac{\mathcal{C}_n}{\gamma_1 \gamma_2 \cdots \gamma_{m-1}} |x - v|^{2m-(n+1)} \right] \\
&= \frac{(2m-n-1)\mathcal{C}_n}{\gamma_1 \gamma_2 \cdots \gamma_{m-1}} (x_j - v_j) |x - v|^{2m-(n+3)} \\
&= \frac{(2m-n-1)\mathcal{C}_n}{\delta_{2-(n-1)} \delta_{4-(n-1)} \cdots \delta_{2(m-1)-(n-1)}} (x_j - v_j) |x - v|^{2m-(n+3)} \\
&= \frac{(2m-n-1)\mathcal{C}_n}{\frac{2(m-1)-(n-1)}{-(n-1)} \alpha_{-(n-1)} \alpha_{2-(n-1)} \cdots \alpha_{2(m-2)-(n-1)}} (x_j - v_j) \\
&\quad \times |x - v|^{2m-(n+3)} \\
&= \frac{\mathcal{C}_n}{\alpha_{2-(n+1)} \alpha_{4-(n+1)} \cdots \alpha_{2(m-1)-(n+1)}} (x_j - v_j) |x - v|^{2m-(n+3)} \\
&= \frac{\mathcal{C}_n}{\beta_1 \beta_2 \cdots \beta_{m-1}} (x_j - v_j) |x - v|^{2m-(n+3)} \\
&= D_m^{(j)}(x, v)
\end{aligned}$$

follows from (2.2), (2.12), (5.6), (5.9) and (5.18).

**Case II:**  $m = \frac{n+1}{2}$  with odd  $n$ .

(5.20)

$$\begin{aligned}
\frac{\partial}{\partial x_j} \mathcal{D}_{\frac{n+1}{2}}(x, v) &= \frac{\partial}{\partial x_j} \left[ \frac{\mathcal{C}_n}{(n-1)\gamma_1 \gamma_2 \cdots \gamma_{\frac{n-1}{2}-1}} \left( \log |x - v| + \frac{1}{n+1} \right) \right] \\
&= \frac{\mathcal{C}_n}{(n-1)\gamma_1 \gamma_2 \cdots \gamma_{\frac{n-1}{2}-1}} (x_j - v_j) |x - v|^{-2} \\
&= \frac{\mathcal{C}_n}{(n-1)\delta_{2-(n-1)} \delta_{4-(n-1)} \cdots \delta_{2(\frac{n-1}{2}-1)-(n-1)}} (x_j - v_j) \\
&\quad \times |x - v|^{-2} \\
&= \frac{\mathcal{C}_n}{(n-1) \left[ \frac{2(\frac{n-1}{2}-1)-(n-1)}{-(n-1)} \alpha_{-(n-1)} \alpha_{2-(n-1)} \cdots \alpha_{2(\frac{n-1}{2}-2)-(n-1)} \right]} \\
&\quad \times (x_j - v_j) |x - v|^{-2} \\
&= \frac{\mathcal{C}_n}{\frac{1}{-(n-1)} \alpha_{2-(n+1)} \alpha_{4-(n+1)} \cdots \alpha_{2(\frac{n-1}{2}-1)-(n+1)} \alpha_{2(\frac{n+1}{2}-1)-(n+1)}} \\
&\quad \times (x_j - v_j) |x - v|^{-2} \\
&= \frac{\mathcal{C}_n}{\beta_1 \beta_2 \cdots \beta_{\frac{n+1}{2}-1}} (x_j - v_j) |x - v|^{-2} \\
&= D_{\frac{n+1}{2}}^{(j)}(x, v)
\end{aligned}$$

follows from (2.2), (2.12), (5.1), (5.6), (5.9) and (5.18).

**Case III:**  $m \geq \frac{n+3}{2}$  with odd  $n$ .

(5.21)

$$\begin{aligned}
\frac{\partial}{\partial x_j} \mathcal{D}_m(x, v) &= \frac{\partial}{\partial x_j} \left\{ \frac{\mathcal{C}_n}{(n-1)\gamma_1\gamma_2 \cdots \gamma_{\frac{n-1}{2}-1} \delta_2 \delta_4 \cdots \delta_{2m-n-1}} |x-v|^{2m-(n+1)} \right. \\
&\quad \times \left[ \log|x-v| + \frac{1}{n+1} - \sum_{t=1}^{m-\frac{n+1}{2}} \left( \frac{1}{2t} + \frac{1}{2t+n-1} \right) \right] \Big\} \\
&= \frac{(2m-n-1)\mathcal{C}_n}{(n-1)\gamma_1\gamma_2 \cdots \gamma_{\frac{n-1}{2}-1} \delta_2 \delta_4 \cdots \delta_{2m-n-1}} (x_j - v_j) |x-v|^{2m-(n+3)} \\
&\quad \times \left[ \log|x-v| + \frac{1}{n+1} - \sum_{t=1}^{m-\frac{n+1}{2}} \left( \frac{1}{2t} + \frac{1}{2t+n-1} \right) \right] \\
&\quad + \frac{\mathcal{C}_n}{(n-1)\gamma_1\gamma_2 \cdots \gamma_{\frac{n-1}{2}-1} \delta_2 \delta_4 \cdots \delta_{2m-n-1}} (x_j - v_j) |x-v|^{2m-(n+3)} \\
&= \frac{(2m-n-1)\mathcal{C}_n}{(n-1)\delta_{2-(n-1)}\delta_{4-(n-1)} \cdots \delta_{2(\frac{n-1}{2}-1)-(n-1)} \delta_2 \delta_4 \cdots \delta_{2m-n-1}} \\
&\quad \times (x_j - v_j) |x-v|^{2m-(n+3)} \left[ \log|x-v| + \frac{1}{n+1} \right. \\
&\quad \left. - \sum_{t=1}^{m-\frac{n+1}{2}} \left( \frac{1}{2t} + \frac{1}{2t+n-1} \right) \right] \\
&\quad + \frac{\mathcal{C}_n}{(n-1)\delta_{2-(n-1)}\delta_{4-(n-1)} \cdots \delta_{2(\frac{n-1}{2}-1)-(n-1)} \delta_2 \delta_4 \cdots \delta_{2m-n-1}} \\
&\quad \times (x_j - v_j) |x-v|^{2m-(n+3)} \\
&= \frac{c_n}{(n+1)\beta_1\beta_2 \cdots \beta_{\frac{n+1}{2}-1} \alpha_2\alpha_4 \cdots \alpha_{2m-n-3}} (x_j - v_j) |x-v|^{2m-(n+3)} \\
&\quad \times \left[ \log|x-v| + \frac{1}{n+1} - \sum_{t=1}^{m-\frac{n+1}{2}} \left( \frac{1}{2t} + \frac{1}{2t+n-1} \right) \right] \\
&\quad + \frac{1}{2m-n-1} \frac{c_n}{(n+1)\beta_1\beta_2 \cdots \beta_{\frac{n+1}{2}-1} \alpha_2\alpha_4 \cdots \alpha_{2m-n-3}} \\
&\quad \times (x_j - v_j) |x-v|^{2m-(n+3)} \\
&= \frac{c_n}{(n+1)\beta_1\beta_2 \cdots \beta_{\frac{n+1}{2}-1} \alpha_2\alpha_4 \cdots \alpha_{2m-n-3}} (x_j - v_j) |x-v|^{2m-(n+3)} \\
&\quad \times \left[ \log|x-v| - \sum_{t=1}^{m-\frac{n+3}{2}} \left( \frac{1}{2t} + \frac{1}{2t+n+1} \right) \right] \\
&= D_m^{(j)}(x, v)
\end{aligned}$$

follows from (2.2), (2.12), (5.1), (5.6), (5.9) and (5.18), where the fourth equality is based on the following calculations (by repeatedly invoking (5.18)):

(5.22)

$$\begin{aligned}
& (n-1)\delta_{2-(n-1)}\delta_{4-(n-1)}\cdots\delta_{2(\frac{n-1}{2}-1)-(n-1)}\delta_2\delta_4\cdots\delta_{2m-n-1} \\
&= 2(n-1)(n+1)\prod_{k=0}^{\frac{n-1}{2}-2}\left[\frac{2(k+1)-(n-1)}{2k-(n-1)}\alpha_{2k-(n-1)}\right]\times\prod_{l=1}^{m-\frac{n+3}{2}}\left[\frac{2l+2}{2l}\alpha_{2l}\right] \\
&= \frac{2m-n-1}{1-n}\left\{(n+1)\prod_{k=1}^{\frac{n-1}{2}-1}\alpha_{2k-(n+1)}\times[-2(n-1)]\times\prod_{l=1}^{m-\frac{n+3}{2}}\alpha_{2l}\right\} \\
&= \frac{2m-n-1}{1-n}\left\{(n+1)\prod_{k=1}^{\frac{n-1}{2}-1}\alpha_{2k-(n+1)}\times[(-2)(-2+n+1)]\times\prod_{l=1}^{m-\frac{n+3}{2}}\alpha_{2l}\right\} \\
&= \frac{2m-n-1}{1-n}\left\{(n+1)\prod_{k=1}^{\frac{n+1}{2}-1}\alpha_{2k-(n+1)}\times\prod_{l=1}^{m-\frac{n+3}{2}}\alpha_{2l}\right\} \\
&= \frac{2m-n-1}{1-n}(n+1)\beta_1\beta_2\cdots\beta_{\frac{n+1}{2}-1}\alpha_2\alpha_4\cdots\alpha_{2m-n-3}
\end{aligned}$$

in which  $-2(n-1) = (-2)(-2+(n+1)) = \alpha_{2(\frac{n+1}{2}-1)-(n+1)} = \beta_{\frac{n+1}{2}-1}$  that has been already used in the fifth equality of (5.20).

By (5.17), we have

$$(5.23) \quad \frac{\partial}{\partial x_j}\mathcal{K}_m(x, v) - K_m^{(j)}(x, v) = \text{S.P.}[D_m^{(j)}](x, v) - \frac{\partial}{\partial x_j}\text{S.P.}[\mathcal{D}_m](x, v)$$

for any  $x, v \in \mathbb{R}^{n+1}$  with  $x \neq v$  and sufficiently large  $|v|$  (in fact, for any  $|v|$ ). By Definition 2.3,  $\frac{\partial}{\partial x_j}\mathcal{K}_m(x, v) - K_m^{(j)}(x, v) = \text{S.P.}[D_m^{(j)}](x, v) - \frac{\partial}{\partial x_j}\text{S.P.}[\mathcal{D}_m](x, v) = 0$ . Then (5.15) follows and the theorem is completed.  $\square$

*Remark 5.5.* In the proofs of the above theorem and Theorem 3.5, we respectively obtain that

$$(5.24) \quad \text{S.P.}[D_m^{(j)}](x, v) = \frac{\partial}{\partial x_j}\text{S.P.}[\mathcal{D}_m](x, v)$$

and

$$(5.25) \quad \text{S.P.}[D_{m-1}^{(j)}](x, v) = \Delta\text{S.P.}[D_m^{(j)}](x, v).$$

Form these identities, it is easy to find some identities on the ultraspherical polynomials  $P_l^{(\lambda)}$  and  $Q_l^{(\lambda)}$ . However, we will not want to pursue these results in this article.

## 6. POLYHARMONIC NEUMANN PROBLEMS IN LIPSCHITZ GRAPHIC DOMAINS

In this section, we will consider the polyharmonic Neumann problems (1.2) in Lipschitz graphic domains as follows

$$(6.1) \quad \begin{cases} \Delta^m u = 0, & \text{in } D, \\ \frac{\partial}{\partial N} \Delta^j u = g_j, & \text{on } \partial D, \end{cases}$$

where  $\nabla(u - \mathcal{M}_1 \tilde{g}_0) \in L^p(D)$  with  $\|\nabla(u - \mathcal{M}_1 \tilde{g}_0)\|_{L^p(D)} \leq C \sum_{j=1}^{m-1} \|g_j\|_{L^p(\partial D, w d\sigma)}$ , the Laplacian  $\Delta = \sum_{k=1}^{n+1} \frac{\partial^2}{\partial x_k^2}$ , the gradient operator  $\nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_{n+1}} \right)$ ,  $D$  is a Lipschitz graphic domain stated as in Theorem 3.5,  $g_0 \in L^p(\partial D)$ ,  $g_j \in L^p(\partial D, w d\sigma)$  for some suitable  $p > 1$ , the  $(p, 2m - 2, \frac{3}{2})$  weight  $w$  on  $\partial D$  is given in Section 3.1,  $\frac{\partial}{\partial N}$  denotes the outward normal derivative,  $\tilde{g}_0$  is related to all the boundary data  $g_j$ ,  $0 \leq j < m$  and  $m \in \mathbb{N}$ .

**Definition 6.1.** Let  $D$  be a Lipschitz domain in  $\mathbb{R}^{n+1}$  with the boundary  $\partial D$ . Set

$$(6.2) \quad \mathcal{M}_j f(X) = \int_{\partial D} \mathcal{K}_j(X, Q) f(Q) d\sigma(Q), \quad X \in D,$$

where  $1 \leq j < \infty$ ,  $\mathcal{K}_j$  is the  $j$ th order polyharmonic fundamental solution,  $d\sigma$  is the surface measure on  $\partial D$ , and  $f \in L^p(\partial D)$  for some suitable  $p$ .  $\mathcal{M}_j f$  is called the  $j$ th-layer  $\mathcal{S}$ -potential of  $f$ .

*Remark 6.2.* It is well known that  $-\mathcal{K}_1$  is the fundamental solution of the Laplacian and  $\mathcal{M}_1$  is the classical single layer potential.

By the properties of polyharmonic fundamental solutions, we have

**Theorem 6.3.** Let  $\{\mathcal{K}_m\}_{m=1}^\infty$  be the sequence of the polyharmonic fundamental solutions, and  $D$  be a Lipschitz graphic domain in  $\mathbb{R}^{n+1}$  with Lipschitz graphic boundary  $\partial D$ , which is the same as in Theorem 3.5, then

- (1) For all  $m \in \mathbb{N}$ ,  $\mathcal{K}_m \in (C^\infty \times C)(D \times \partial D)$ , the non-tangential boundary value

$$\lim_{\substack{X \rightarrow P \\ X \in \Gamma_\gamma(P), Q \in \partial D}} \mathcal{K}_m(X, Q) = \mathcal{K}_m(P, Q)$$

exists for all  $P \in \partial D$  and  $P \neq Q \in \partial D$ ;  $\mathcal{K}_m(\cdot, P)$  can be continuously extended to  $\overline{D} \setminus \{P\}$  for any fixed  $P \in \partial D$ ;

- (2) For  $m \geq 2$ ,

$$|\mathcal{K}_m(X, Q)| \leq M \frac{1}{(1 + |Q|^2)^{\frac{n+1+\epsilon}{2}}}$$

for any  $(X, Q) \in D_c \times \{Q \in \partial D : |Q| > T\}$ , where  $0 < \epsilon < 1$ ,  $D_c$  is any compact subset of  $\overline{D}$ ,  $T$  is a sufficiently large positive real number and  $M$  denotes some positive constant depending only on  $\epsilon$ ,  $D_c$  and  $T$ ;

- (3)  $\Delta_X \mathcal{K}_1(X, Y) = \Delta_Y \mathcal{K}_1(X, Y) = 0$  and  $\Delta_X \mathcal{K}_m(X, Y) = \Delta_Y \mathcal{K}_m(X, Y) = \mathcal{K}_{m-1}(X, Y)$  for  $m > 1$ ,  $X, Y \in \mathbb{R}^{n+1} \setminus \{0\}$  and  $X \neq Y$ , where  $\Delta_X = \sum_{j=1}^{n+1} \frac{\partial}{\partial x_j}$  and  $\Delta_Y = \sum_{j=1}^{n+1} \frac{\partial}{\partial y_j}$ ;

- (4) The non-tangential limit

$$(6.3) \quad \lim_{\substack{X \rightarrow P \\ X \in \Gamma_\gamma(P)}} \left\langle \nabla \left( \int_{\partial D} \mathcal{K}_1(X, Q) f(Q) d\sigma(Q) \right), n_P \right\rangle = -\frac{1}{2} f(P) + T^* f(P),$$

for any  $f \in L^p(\partial D)$ ,  $1 \leq p < \infty$ ;

(5) *The non-tangential limit*

$$(6.4) \quad \lim_{\substack{X \rightarrow P \\ X \in \Gamma_\gamma(P)}} \left\langle \nabla \left( \int_{\partial D} \mathcal{K}_m(X, Q) f(Q) d\sigma(Q) \right), n_P \right\rangle = -K_m^* f(P),$$

for any  $m \geq 2$  and  $f \in L^p(\partial D)$ ,  $1 \leq p \leq \infty$ , where

$$(6.5) \quad K_m^* f(P) = \int_{\partial D} \langle K_m(Q, P), n_P \rangle f(Q) d\sigma(Q)$$

which is the adjoint operator of  $K_m$ .

*Proof.* It is similar to Theorem 3.5 by invoking Lemma 5.1 and Theorem 5.4.  $\square$

*Remark 6.4.* The operator  $K_m^*$  has the same boundedness as the operator  $K_m$  does. For instance, it is also bounded from  $L^p(\partial D, w d\sigma)$  to  $L^p(\partial D)$  for any  $w \in W^{p, 2m-2, \frac{1}{2}}(\partial D)$  and  $1 \leq p \leq \infty$ . The details can be seen in the following Theorem 6.8 in the following Section 6.1.

**Theorem 6.5.** *Let  $\{\mathcal{K}_m\}_{m=1}^\infty$  be the sequence of the polyharmonic fundamental solutions, and  $E$  be a simply connected unbounded domain in  $\mathbb{R}^{n+1}$  with smooth boundless boundary  $\partial E$ . Then for any  $m > 1$  and  $f \in L^p(\partial E)$ ,  $p \geq 1$ ,*

$$(6.6) \quad \Delta \left( \int_{\partial E} \mathcal{K}_m(X, Q) f(Q) d\sigma(Q) \right) = \int_{\partial E} \mathcal{K}_{m-1}(X, Q) f(Q) d\sigma(Q),$$

where  $X \in \mathbb{R}^{n+1} \setminus \partial E$ , namely,

$$(6.7) \quad \Delta \mathcal{M}_m f(X) = \mathcal{M}_{m-1} f(X), \quad X \in \mathbb{R}^{n+1} \setminus \partial E.$$

*Proof.* It is similar to Theorem 4.3 by using the analogues of Lemma 4.1, Corollary 4.2 and the claim (3) in the last theorem.  $\square$

*Remark 6.6.* As Remark 4.4 stated, the above theorem also holds in the case of replacing the smooth domain  $E$  by the Lipschitz graphic domain  $D$  given in Theorem 3.5.

By the last two theorems, Lemma 3.4 and the results in the following Section 6.1, we can solve the polyharmonic Neumann problems in Lipschitz domains as follows.

**Theorem 6.7.** *Let  $\{\mathcal{K}_m\}_{m=1}^\infty$  be the sequence of the polyharmonic fundamental solutions, and  $D$  be a Lipschitz graphic domain in  $\mathbb{R}^{n+1}$  with Lipschitz graphic boundary  $\partial D$  as in Theorem 3.5, then for any  $m > 1$ , there exists  $\varepsilon = \varepsilon(D) > 0$  such that the PHN problem (1.2) with the data  $g_0 \in L^p(\partial D)$ ,  $g_j \in L^p(\partial D, w d\sigma)$  with  $w \in W^{p, 2m-2, \frac{3}{2}}(\partial D)$ ,  $1 \leq j < m$ ,  $1 < p < 2 + \varepsilon$ , is solvable and a solution is given by*

$$(6.8) \quad \begin{aligned} u(X) &= \sum_{j=1}^m \int_{\partial D} \mathcal{K}_j(X, Q) \tilde{g}_{j-1}(Q) d\sigma(Q), \\ &= \sum_{j=1}^m \mathcal{M}_j \tilde{g}_{j-1}(X), \quad X \in D, \end{aligned}$$

where

$$(6.9) \quad \tilde{g}_{m-1} = \left( -\frac{1}{2}I + T^* \right)^{-1} g_{m-1}$$

and

$$(6.10) \quad \tilde{g}_l = \left( -\frac{1}{2}I + T^* \right)^{-1} \left( g_l + \sum_{j=l+2}^m K_{j-l}^* \tilde{g}_{j-1} \right)$$

with  $0 \leq l \leq m-2$ , which satisfying the following estimate

$$(6.11) \quad \|\nabla(u - \mathcal{M}_1 \tilde{g}_0)\|_{L^p(D)} \leq C \sum_{j=1}^{m-1} \|g_j\|_{L^p(\partial D, w d\sigma)}.$$

Under this estimate, the solution (6.8) with (6.9) and (6.10) is unique up to a constant. Furthermore, if  $w \in \mathcal{W}^{p, 2m-1, \frac{3}{2}}(\partial D) (\subset \mathcal{W}^{p, 2m-1, \frac{3}{2}}(\partial D))$ , then the above solution also satisfies the following estimate

$$(6.12) \quad \|u - \mathcal{M}_1 \tilde{g}_0\|_{L^p(D)} \leq C \sum_{j=1}^{m-1} \|g_j\|_{L^p(\partial D, w d\sigma)},$$

and is unique under the last estimate.

*Proof.* It is similar to Theorem 4.5 by lemma 3.4, and Theorems 6.3, 6.8, 6.11 and 6.12 below.  $\square$

**6.1.  $L^p$  bounded properties of operators  $K_m^*$  and multi-layer  $\mathcal{S}$ -potentials  $\mathcal{M}_j$  and their gradients.** In this section, we study the  $L^p$  bounded properties of the operators  $K_m^*$  given in (6.5) and the multi-layer  $\mathcal{S}$ -potentials  $\mathcal{M}_j$  defined by (6.2) and their gradients, which are very significant for the solving program to the PHN and PHR problems (i.e., (1.2) and (1.3)) in this paper. More precisely, we have

**Theorem 6.8.** *Let the Lipschitz domain  $D$  and the operators  $K_m^*$ ,  $m \geq 2$ , be the same as before,  $w \in \mathcal{W}^{p, 2m-2, \frac{1}{2}}(\partial D)$ ,  $1 \leq p < \infty$ , then*

$$(6.13) \quad \|K_m^* f\|_{L^p(\partial D)} \leq C \|f\|_{L^p(\partial D, w d\sigma)}$$

for any  $f \in L^p(\partial D, w d\sigma)$ , where  $C$  is a constant depending only on  $m, n, p$  and  $d_0 = \text{dist}(0, \partial D)$ . That is,  $K_m^*$ ,  $m \geq 2$ , are bounded from  $L^p(\partial D, w d\sigma)$  to  $L^p(\partial D)$  for any  $w \in \mathcal{W}^{p, 2m-2, \frac{1}{2}}(\partial D)$  with  $1 \leq p < \infty$ .

*Proof.* It is similar to the argument of Theorem 3.11.  $\square$

**Theorem 6.9.** *Let the Lipschitz domain  $D$  and operators  $\mathcal{M}_j$ ,  $j \geq 2$ , be the same as before,  $w \in \mathcal{W}^{p, 2j-1, \frac{1}{2}}(\partial D)$ ,  $1 \leq p < \infty$ , then*

$$(6.14) \quad \|\mathcal{M}_j f\|_{L^p(\partial D)} \leq C \|f\|_{L^p(\partial D, w d\sigma)}$$

for any  $f \in L^p(\partial D, w d\sigma)$ , where  $C$  is a constant depending only on  $m, n, p$  and  $d_0$ . That is,  $\mathcal{M}_j$ ,  $j \geq 2$ , are bounded from  $L^p(\partial D, w d\sigma)$  to  $L^p(\partial D)$  for any  $w \in \mathcal{W}^{p, 2j-1, \frac{1}{2}}(\partial D)$  with  $1 \leq p < \infty$ .

*Proof.* It is similar to Theorem 3.11.  $\square$

**Theorem 6.10.** *Let the Lipschitz domain  $D$  and operators  $\mathcal{M}_j$ ,  $j \geq 2$ , be the same as before,  $w \in \mathcal{W}^{p, 2j-2, \frac{1}{2}}(\partial D)$ ,  $1 \leq p < \infty$ , then*

$$(6.15) \quad \|\nabla \mathcal{M}_j f\|_{L^p(\partial D)} \leq C \|f\|_{L^p(\partial D, w d\sigma)}$$



for any  $f \in L^p(\partial D, w d\sigma)$ , where  $C$  is a constant depending only on  $m, n, p$  and  $d_0$ . That is,  $\nabla \mathcal{M}_j$ ,  $j \geq 2$ , are bounded from  $L^p(\partial D, w d\sigma)$  to  $L^p(\partial D)$  for any  $w \in \mathcal{W}^{p, 2j-2, \frac{1}{2}}(\partial D)$  with  $1 \leq p < \infty$ .

*Proof.* It is similar to the argument of Theorem 3.11 by using Theorem 5.4.  $\square$

**Theorem 6.11.** *Let the Lipschitz domain  $D$  and operators  $\mathcal{M}_j$ ,  $j \geq 2$ , be the same as before,  $w \in \mathcal{W}^{p, 2j-1, \frac{3}{2}}(\partial D)$ ,  $1 \leq p < \infty$ , then*

$$(6.16) \quad \|\mathcal{M}_j f\|_{L^p(D)} \leq C \|f\|_{L^p(\partial D, w d\sigma)}$$

for any  $f \in L^p(\partial D, w d\sigma)$ , where  $C$  is a constant depending only on  $m, n, p$  and  $d_0$ . That is,  $\mathcal{M}_j$ ,  $j \geq 2$ , are bounded from  $L^p(\partial D, w d\sigma)$  to  $L^p(D)$  for any  $w \in \mathcal{W}^{p, 2j-1, \frac{3}{2}}(\partial D)$  with  $1 \leq p < \infty$ .

*Proof.* It is similar to Theorem 3.12.  $\square$

**Theorem 6.12.** *Let the Lipschitz domain  $D$  and operators  $\mathcal{M}_j$ ,  $j \geq 2$ , be the same as before,  $w \in \mathcal{W}^{p, 2j-2, \frac{3}{2}}(\partial D)$ ,  $1 \leq p < \infty$ , then*

$$(6.17) \quad \|\nabla \mathcal{M}_j f\|_{L^p(D)} \leq C \|f\|_{L^p(\partial D, w d\sigma)}$$

for any  $f \in L^p(\partial D, w d\sigma)$ , where  $C$  is a constant depending only on  $m, n, p$  and  $d_0$ . That is,  $\nabla \mathcal{M}_j$ ,  $j \geq 2$ , are bounded from  $L^p(\partial D, w d\sigma)$  to  $L^p(D)$  for any  $w \in \mathcal{W}^{p, 2j-2, \frac{3}{2}}(\partial D)$  with  $1 \leq p < \infty$ .

*Proof.* It is similar to the argument of Theorem 3.12 by invoking Theorem 5.4.  $\square$

## 7. REGULARITY OF POLYHARMONIC DIRICHLET PROBLEMS IN LIPSCHITZ GRAPHIC DOMAINS

In this section, we will consider the polyharmonic regularity problems (1.3) in Lipschitz domains as follows

$$(7.1) \quad \begin{cases} \Delta^m u = 0, & \text{in } D, \\ \Delta^j u = h_j, & \text{on } \partial D, \end{cases}$$

where  $\nabla(u - \mathcal{M}_1 \tilde{h}_0) \in L^p(D)$  with  $\|\nabla(u - \mathcal{M}_1 \tilde{h}_0)\|_{L^p(D)} \leq C \sum_{j=1}^{m-1} \|h_j\|_{L_1^p(\partial D, w d\sigma)}$ , the Laplacian  $\Delta = \sum_{k=1}^{n+1} \frac{\partial^2}{\partial x_k^2}$ , the gradient operator  $\nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_{n+1}} \right)$ ,  $D$  is a Lipschitz graphic domain stated as Theorem 3.5,  $h_0 \in L_1^p(\partial D)$ ,  $h_j \in L_1^p(\partial D, w d\sigma)$  for some suitable  $p > 1$ , the  $(p, 2m-1, \frac{3}{2})$  weight  $w$  on  $\partial D$  is given in Section 3.1,  $\tilde{h}_0$  is related to all the boundary data  $h_j$ ,  $0 \leq j < m$  and  $m \in \mathbb{N}$ .

Once more, due to Dahlberg, Kenig and Verchota et al., we have

**Lemma 7.1** ([12, 53]). *There exists  $\varepsilon = \varepsilon(D) > 0$  such that  $\mathcal{M}_1$  is an invertible mapping from  $L^p(\partial D)$  onto  $L_1^p(\partial D)$ ,  $1 < p < 2 + \varepsilon$ , where  $L_1^p(\partial D) = \{f \in L^p(\partial D) : \nabla_T f \text{ exist a.e. on } \partial D, \text{ and } |\nabla_T f| \in L^p(\partial D)\}$  with the norm  $\|f\|_{L_1^p(\partial D)} = \|f\|_{L^p(\partial D)} + \|\nabla_T f\|_{L^p(\partial D)}$  in which  $\nabla_T$  is the tangential gradient.*

**Theorem 7.2.** *Let  $\{\mathcal{K}_m\}_{m=1}^\infty$  be the sequence of the polyharmonic fundamental solutions, and  $D$  be a Lipschitz graphic domain in  $\mathbb{R}^{n+1}$  with Lipschitz graphic boundary  $\partial D$  as in Theorem 3.5, then for any  $m > 1$ , there exists  $\varepsilon = \varepsilon(D) > 0$  such that the PHR problem (1.3) with the data  $h_0 \in L_1^p(\partial D)$ ,  $h_j \in L_1^p(\partial D, w d\sigma)$*

with  $w \in \mathcal{W}^{p, 2m-1, \frac{3}{2}}(\partial D)$ ,  $1 \leq j < m$ ,  $1 < p < 2 + \varepsilon$ , is solvable and a solution is given by

$$(7.2) \quad \begin{aligned} u(X) &= \sum_{j=1}^m \int_{\partial D} \mathcal{K}_j(X, Q) \tilde{h}_{j-1}(Q) d\sigma(Q), \\ &= \sum_{j=1}^m \mathcal{M}_j \tilde{h}_{j-1}(X), \quad X \in D, \end{aligned}$$

where

$$(7.3) \quad \tilde{h}_{m-1} = \mathcal{M}_1^{-1} h_{m-1}$$

and

$$(7.4) \quad \tilde{h}_l = \mathcal{M}_1^{-1} \left( h_l - \sum_{j=l+2}^m \mathcal{M}_{j-l} \tilde{h}_{j-1} \right)$$

with  $0 \leq l \leq m-2$ , which satisfying the following estimate

$$(7.5) \quad \|\nabla(u - \mathcal{M}_1 \tilde{h}_0)\|_{L^p(D)} \leq C \sum_{j=1}^{m-1} \|h_j\|_{L_1^p(\partial D, w d\sigma)}.$$

Under this estimate, the solution (7.2) with (7.3) and (7.4) is unique up to a constant. Furthermore, the above solution also satisfies the following estimate

$$(7.6) \quad \|u - \mathcal{M}_1 \tilde{h}_0\|_{L^p(D)} \leq C \sum_{j=1}^{m-1} \|h_j\|_{L_1^p(\partial D, w d\sigma)}.$$

and is unique under the last estimate.

*Proof.* It is similar to Theorem 4.5 by using Lemma 7.1, Theorems 6.9-6.12 and 7.3 below.  $\square$

**7.1. Regularity of multi-layer  $\mathcal{S}$ -potentials  $\mathcal{M}_j$ .** In this section, we study the regularity of the multi-layer  $\mathcal{S}$ -potentials  $\mathcal{M}_j$ , which are very significant for the solving program to the PHR problems (1.3) in this paper. More precisely, we have

**Theorem 7.3.** *Let the Lipschitz domain  $D$  and operators  $\mathcal{M}_j$ ,  $j \geq 2$ , be the same as before,  $w \in \mathcal{W}^{p, 2j-2, \frac{1}{2}}(\partial D)$ ,  $1 \leq p < \infty$ , then*

$$(7.7) \quad \|\nabla_T \mathcal{M}_j f\|_{L^p(\partial D)} \leq C \|f\|_{L^p(\partial D, w d\sigma)}$$

for any  $f \in L^p(\partial D, w d\sigma)$ , where  $\nabla_T$  denotes the tangential gradient,  $C$  is a constant depending only on  $m, n, p$  and  $d_0$ . So  $\mathcal{M}_j$ ,  $j \geq 2$ , are bounded from  $L^p(\partial D, w d\sigma)$  to  $L_1^p(\partial D)$  for any  $w \in \mathcal{W}^{p, 2j-1, \frac{1}{2}}(\partial D)$  with  $1 \leq p < \infty$ .

*Proof.* It is similar to the argument of Theorem 3.11, or directly follows from Theorems 6.8 and 6.10 by the following fact

$$\begin{aligned}
 (7.8) \quad \|\nabla_T \mathcal{M}_j f\|_{L^p(\partial D)} &= \left\| \nabla \mathcal{M}_j f - \left( \frac{\partial}{\partial N} \mathcal{M}_j f \right) \cdot n \right\|_{L^p(\partial D)} \\
 &\leq 2^{p-1} \left( \|\nabla \mathcal{M}_j f\|_{L^p(\partial D)} + \left\| \frac{\partial}{\partial N} \mathcal{M}_j f \right\|_{L^p(\partial D)} \right) \\
 &= 2^{p-1} \left( \|\nabla \mathcal{M}_j f\|_{L^p(\partial D)} + \|K_j^* f\|_{L^p(\partial D)} \right) \\
 &\leq C \|f\|_{L^p(\partial D, w d\sigma)}
 \end{aligned}$$

since  $\nabla \mathcal{M}_j f = \nabla_T \mathcal{M}_j f \oplus \left( \frac{\partial}{\partial N} \mathcal{M}_j f \right) \cdot n$ , where  $\oplus$  denotes the operation of direct sum, and  $n$  is the outward unit normal vector.  $\square$

*Remark 7.4.* It must be noteworthy, using the facts in Remark 3.10, that all the results in Sections 3.1-7 hold with the weight spaces  $\mathcal{W}^{p,l,\frac{1}{2}}(\partial D)$  and  $\mathcal{W}^{p,l,\frac{3}{2}}(\partial D)$  replaced by  $\mathcal{W}^{p,l,\epsilon}(\partial D)$  and  $\mathcal{W}^{p,l,1+\epsilon}(\partial D)$  for any  $0 < \epsilon < 1$ .

## 8. BOUNDED LIPSCHITZ DOMAINS

In this section, we mainly consider the corresponding polyharmonic Dirichlet, Neumann, and regularity problems in  $L^p$  in bounded Lipschitz domains. Throughout this section, the higher order conjugate Poisson and Poisson kernels  $K_m^{(j)} = D_m^{(j)}$ , and the polyharmonic fundamental solutions  $\mathcal{K}_m = \mathcal{D}_m$ ,  $1 \leq j \leq n+1$ ,  $m \in \mathbb{N}$ . In other words, here  $\text{S.P.}[K_m^{(j)}] \equiv 0$  and  $\text{S.P.}[\mathcal{K}_m] \equiv 0$  for any  $1 \leq j \leq n+1$  and  $m \in \mathbb{N}$ .

In the same way, due to Dahlberg, Kenig and Verchota et al., we have

**Lemma 8.1** ([12, 53]). *There exists  $\varepsilon = \varepsilon(D) > 0$  such that  $\frac{1}{2}I - T^*$  is an invertible mapping from  $L_0^p(\partial D)$  onto  $L_0^p(\partial D)$ ,  $1 < p < 2 + \varepsilon$ , where  $L_0^p(\partial D) = \{f \in L^p(\partial D) : \int_{\partial D} f d\sigma = 0\}$ .*

As some preliminaries, we firstly establish some lemmas as follows.

**Lemma 8.2.** *Let  $D$  be a bounded Lipschitz domain,  $D_m = (D_m^{(1)}, \dots, D_m^{(n+1)})$  in which  $D_m^{(j)}$  are defined as in Lemma 2.2, then there exists a constant  $C = C(m, n, D)$  such that*

$$(8.1) \quad \sup_{Q \in \partial D} \left( \int_{\partial D} |\langle D_m(Q, P), n_P \rangle| d\sigma(P) \right) < C$$

and

$$(8.2) \quad \sup_{Q \in \partial D} \left( \int_{\partial D} |\langle D_m(Q, P), n_Q \rangle| d\sigma(P) \right) < C$$

for any  $m \geq 2$ , where  $n_P$  and  $n_Q$  are the unit outward normal vectors respectively at  $P$  and  $Q$  on  $\partial D$ .

*Proof.* At first, we observe that

$$(8.3) \quad |\langle D_m(Q, P), n \rangle| \leq C_{m,n} |P - Q|^{2m-(n+2)} (1 + |\log |P - Q||).$$

So it is sufficient to verify (8.1). By the definition of bounded Lipschitz domain, set  $\{L_1, \dots, L_s\}$  be a finite cover of circular coordinate cylinders on  $\partial D$  centered

respectively at  $Q_j$ ,  $1 \leq j \leq s$  whose bases have positive distances from  $\partial D$ . That is, there exists a Lipschitz function  $\varphi_j : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $1 \leq j \leq s$  such that

- (i):  $|\varphi_j(\underline{x}) - \varphi_j(\underline{y})| \leq \mathcal{L}_j |\underline{x} - \underline{y}|$  for any  $\underline{x}, \underline{y} \in \mathbb{R}^n$  with  $0 < \mathcal{L}_j < \infty$ ;
- (ii):  $L_j \cap D = \{(\underline{x}, x_{n+1}) : x_{n+1} > \varphi_j(\underline{x})\}$ ;
- (iii):  $L \cap \partial D = \{(\underline{x}, x_{n+1}) : x_{n+1} = \varphi_j(\underline{x})\}$ ;
- (iv):  $Q_j = (\underline{Q}, \varphi_j(\underline{Q}))$ ,

where  $\underline{x} = \{x_1, \dots, x_n\} \in \mathbb{R}^n$ . Let  $\mathcal{L} = \max_{1 \leq j \leq s} \mathcal{L}_j$ ,  $\mathcal{L}$  is usually called the Lipschitz constant (or Lipschitz character). By a rearrangement, we can assume that all  $L_j$  are adjacent with each other in turn.

Denote that  $d_j = \text{dist}\{Q_j, \partial(L_j \cap \partial D)\}$ ,  $1 \leq j \leq s$ . In the coordinate system associated with  $(L_j, Q_j)$ , define the projection  $\pi_j : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$  with  $\pi_j(\underline{x}, x_{n+1}) = \underline{x}$ . Let  $U_j = \pi_j(D)$  and  $\rho_j = \max_{\underline{x} \in \partial U_j} |\underline{x} - \underline{Q}|$ . Set  $d = \min_j d_j$  and  $\rho = \max_j \rho_j$ .

To do prove (8.1), let  $Q \in \partial D$  be temporarily fixed. Then  $Q \in L_{j_0} \cap \partial D$  for some  $1 < j_0 < s$ , or possibly  $Q \in L_{j'_0} \cap \partial D$  with  $|j'_0 - j_0| = 1$ . In fact, with respect to the latter case,  $Q \in L_{j_0} \cap L_{j'_0} (\neq \emptyset)$ , and in the following argument, we only consider the latter case, so does the former case. Furthermore, it is easy to find that  $\pi_{j_0}(B(Q, \frac{d}{2}) \cap L_{j_0} \cap \partial D) \subset B_{j_0}(\underline{Q}, \rho)$  and  $\pi_{j'_0}(B(Q, \frac{d}{2}) \cap L_{j'_0} \cap \partial D) \subset B_{j'_0}(\underline{Q}, \rho)$ .

With the above preliminaries, by (8.3), we have

(8.4)

$$\begin{aligned}
\int_{\partial D} |\langle D_m(Q, P), n_Q \rangle| d\sigma(P) &\leq C_{m,n} \int_{\partial D} |P - Q|^{2m-(n+2)} (1 + |\log |P - Q||) d\sigma(P) \\
&\leq C_{m,n, \text{diam}(D)} \int_{\partial D} |P - Q|^{2m-(n+2)-\eta} d\sigma(P) \\
&\leq C_{m,n, \text{diam}(D)} \int_{\partial D} \frac{1}{|P - Q|^{(n-2)+\eta}} d\sigma(P) \quad (\text{since } m \geq 2) \\
&= C_{m,n, \text{diam}(D)} \left[ \int_{\partial D \cap B(Q, \frac{d}{2})} \frac{1}{|P - Q|^{(n-2)+\eta}} d\sigma(P) \right. \\
&\quad \left. + \int_{\partial D \setminus B(Q, \frac{d}{2})} \frac{1}{|P - Q|^{(n-2)+\eta}} d\sigma(P) \right] \\
&\leq C_{m,n, \text{diam}(D)} \left[ \int_{\partial D \cap B(Q, \frac{d}{2})} \frac{1}{|P - Q|^{(n-2)+\eta}} d\sigma(P) \right. \\
&\quad \left. + \left(\frac{2}{d}\right)^{n-2+\eta} \int_{\partial D \setminus B(Q, \frac{d}{2})} d\sigma(P) \right] \\
&\leq C_{m,n, \text{diam}(D)} \left[ \int_{\partial D \cap B(Q, \frac{d}{2})} \frac{1}{|P - Q|^{(n-2)+\eta}} d\sigma(P) \right. \\
&\quad \left. + \left(\frac{2}{d}\right)^{n-2+\eta} \sigma(\partial D) \right]
\end{aligned}$$

in which

$$\begin{aligned}
(8.5) \quad &\int_{\partial D \cap B(Q, \frac{d}{2})} \frac{1}{|P - Q|^{(n-2)+\eta}} d\sigma(P) \\
&\leq \int_{\partial D \cap L_{j_0} \cap B(Q, \frac{d}{2})} \frac{1}{|P - Q|^{(n-2)+\eta}} d\sigma(P)
\end{aligned}$$

$$\begin{aligned}
& + \int_{\partial D \cap L_{j'_0} \cap B(Q, \frac{d}{2})} \frac{1}{|P - Q|^{(n-2)+\eta}} d\sigma(P) \\
& = \int_{\pi_{j_0}(\partial D \cap L_{j_0} \cap B(Q, \frac{d}{2}))} \frac{\sqrt{1 + |\nabla \varphi_{j_0}(\underline{x})|^2}}{(|\underline{x} - \underline{x}_Q|^2 + |\varphi_{j_0}(\underline{x}) - \varphi_{j_0}(\underline{x}_Q)|^2)^{\frac{(n-2)+\eta}{2}}} d\underline{x} \\
& \quad + \int_{\pi_{j'_0}(\partial D \cap L_{j'_0} \cap B(Q, \frac{d}{2}))} \frac{\sqrt{1 + |\nabla \varphi_{j'_0}(\underline{x})|^2}}{(|\underline{x} - \underline{x}_Q|^2 + |\varphi_{j'_0}(\underline{x}) - \varphi_{j'_0}(\underline{x}_Q)|^2)^{\frac{(n-2)+\eta}{2}}} d\underline{x} \\
& \leq \sqrt{1 + \mathcal{L}^2} \left[ \int_{B_{j_0}(\underline{Q}, \rho)} \frac{1}{|\underline{x} - \underline{x}_Q|^{(n-2)+\eta}} d\underline{x} + \int_{B_{j'_0}(\underline{Q}, \rho)} \frac{1}{|\underline{x} - \underline{x}_Q|^{(n-2)+\eta}} d\underline{x} \right] \\
& \leq \sqrt{1 + \mathcal{L}^2} \left[ \int_{B_{j_0}(\underline{x}_Q, 2\rho)} \frac{1}{|\underline{x} - \underline{x}_Q|^{(n-2)+\eta}} d\underline{x} + \int_{B_{j'_0}(\underline{x}_Q, 2\rho)} \frac{1}{|\underline{x} - \underline{x}_Q|^{(n-2)+\eta}} d\underline{x} \right] \\
& \leq 2\sqrt{1 + \mathcal{L}^2} \int_0^{2\rho} \int_{S^{n-1}} \frac{1}{r^{(n-2)+\eta}} r^{n-1} dr d\sigma(\omega) \\
& = \frac{2}{2-\eta} (2\rho)^{2-\eta} \sqrt{1 + \mathcal{L}^2} \sigma(S^{n-1}),
\end{aligned}$$

where  $0 < \eta < 1$  which can be arbitrary selected, the fact  $\lim_{|P-Q| \rightarrow 0} |P - Q|^\eta \log |P - Q| = 0$  has been used in the second inequality of (8.4); whereas in the third inequality in (8.5), we have used the fact that  $\underline{x}, \underline{x}_Q \in B_{j_0}(\underline{Q}, \rho) \cup B_{j'_0}(\underline{Q}, \rho)$  implies  $\underline{x} \in B_{j_0}(\underline{x}_Q, 2\rho) \cup B_{j'_0}(\underline{x}_Q, 2\rho)$ .

Therefore, by (8.4) and (8.5), we have

$$\begin{aligned}
(8.6) \quad \int_{\partial D} |\langle D_m(Q, P), n_Q \rangle| d\sigma(P) & \leq C_{m,n,\text{diam} D} \left[ \frac{2}{2-\eta} (2\rho)^{2-\eta} \sqrt{1 + \mathcal{L}^2} \sigma(S^{n-1}) \right. \\
& \quad \left. + \left( \frac{2}{d} \right)^{n-2+\eta} \sigma(\partial D) \right].
\end{aligned}$$

Denote

$$(8.7) \quad C(m, n, D) = C_{m,n,\text{diam} D} \left[ \frac{2}{2-\eta} (2\rho)^{2-\eta} \sqrt{1 + \mathcal{L}^2} \sigma(S^{n-1}) + \left( \frac{2}{d} \right)^{n-2+\eta} \sigma(\partial D) \right],$$

which depends only on  $m, n$  and  $D$ , then (8.1) follows from (8.6) since  $Q \in \partial D$  is arbitrarily chosen.. Thus the lemma is completed.  $\square$

**Lemma 8.3.** *Let  $D$  be a bounded Lipschitz domain,  $D_m = (D_m^{(1)}, \dots, D_m^{(n+1)})$  in which  $D_m^{(j)}$  are defined as in Lemma 2.2, then there exists a constant  $C = C(m, n, D)$  such that*

$$(8.8) \quad \sup_{X \in D} \left( \int_{\partial D} |\langle D_m(X, P), n_P \rangle| d\sigma(P) \right) < C$$

and

$$(8.9) \quad \sup_{X \in D} \left( \int_{\partial D} |\langle D_m(X, P), n_Q \rangle| d\sigma(P) \right) < C$$

for any  $m \geq 2$ , where  $n_P$  and  $n_Q$  are the outward unit normal vectors respectively at  $P$  and  $Q$  on  $\partial D$ .

*Proof.* It is similar to Lemma 8.2.  $\square$

*Remark 8.4.* Let  $D$  and  $D_m$  be as above, by the above two lemmas or a direct argument, in fact, there exists a constant  $C = C(m, n, D)$  such that

$$(8.10) \quad \sup_{X \in \overline{D}} \left( \int_{\partial D} |\langle D_m(X, P), n_P \rangle| d\sigma(P) \right) < C$$

and

$$(8.11) \quad \sup_{X \in \overline{D}} \left( \int_{\partial D} |\langle D_m(X, P), n_Q \rangle| d\sigma(P) \right) < C$$

for any  $m \geq 2$ , where  $n_P$  and  $n_Q$  are the outward unit normal vectors respectively at  $P$  and  $Q$  on  $\partial D$ .

With  $D_m$  replaced by  $\mathcal{D}_m$ , we also have

**Lemma 8.5.** *Let  $D$  be a bounded Lipschitz domain,  $\mathcal{D}_m$  are defined as in Lemma 5.1, then there exists a constant  $C = C(m, n, D)$  such that*

$$(8.12) \quad \sup_{Q \in \partial D} \left( \int_{\partial D} |\mathcal{D}_m(Q, P)| d\sigma(P) \right) < C$$

for any  $m \geq 2$ .

*Proof.* It is similar to Lemma 8.2.  $\square$

**Lemma 8.6.** *Let  $D$  be a bounded Lipschitz domain,  $\mathcal{D}_m$  are defined as in Lemma 5.1, then there exists a constant  $C = C(m, n, D)$  such that*

$$(8.13) \quad \sup_{X \in D} \left( \int_{\partial D} |\mathcal{D}_m(X, P)| d\sigma(P) \right) < C$$

for any  $m \geq 2$ .

*Proof.* It is similar to Lemma 8.5.  $\square$

*Remark 8.7.* Let  $D$  and  $\mathcal{D}_m$  be as above, by Lemmas 8.5 and 8.6 or a direct argument, in fact, we have that there exists a constant  $C = C(m, n, D)$  such that

$$(8.14) \quad \sup_{X \in \overline{D}} \left( \int_{\partial D} |\mathcal{D}_m(X, P)| d\sigma(P) \right) < C$$

for any  $m \geq 2$ .

Furthermore, we have

**Lemma 8.8.** *Let  $D$  be a bounded Lipschitz domain,  $\mathcal{D}_m$  are defined as in Lemma 5.1, then there exists a constant  $C = C(m, n, D)$  such that*

$$(8.15) \quad \sup_{X \in \overline{D}} \left( \int_{\partial D} |\nabla \mathcal{D}_m(X, P)| d\sigma(P) \right) < C$$

for any  $m \geq 2$ .

*Proof.* By (5.17),  $\nabla \mathcal{D}_m = D_m$ . So it is similar to Lemma 8.2 as Remark 8.4 states.  $\square$

*Remark 8.9.* By observing the argument of Lemma 8.2, it is easy to find that Lemmas 8.5 and 8.6, as well as (8.14) in Remark 8.7 also holds when  $m = 1$ .

In terms of above lemmas, we can obtain some bounded properties in  $L^p$  for the operators  $K_m^*$ ,  $K_m$ ,  $M_j$ ,  $\mathcal{M}_j$  and  $\nabla \mathcal{M}_j$  and so on, which are important in the approach to solve the polyharmonic BVPs (1.1)-(1.3) in the case of bounded Lipschitz domains of this section.

**Theorem 8.10.** *Let  $D$  be a bounded Lipschitz domain,  $K_m^*$ ,  $m \geq 2$  be as in Theorem 6.3, then*

$$(8.16) \quad \|K_m^* f\|_{L^p(\partial D)} \leq C \|f\|_{L^p(\partial D)}$$

for any  $f \in L^p(\partial D)$ ,  $1 \leq p \leq \infty$ . Furthermore, if

$$(8.17) \quad \int_{\partial D} \mathcal{N}_{m-1}(Q) f(Q) d\sigma(Q) = 0,$$

then

$$(8.18) \quad \int_{\partial D} K_m^* f(P) d\sigma(P) = 0,$$

where  $\mathcal{N}_{m-1}$  is the  $(m-1)$ -th order Newtonian potential on  $D$  defined as follows

$$(8.19) \quad \mathcal{N}_{m-1}(Y) = \int_D \mathcal{D}_{m-1}(X, Y) dX, \quad Y \in \mathbb{R}^{n+1}.$$

*Remark 8.11.* The classical Newtonian potential is referred to [29].

*Proof.* At first, it is easy to verify (8.16). In fact, by (8.1),  $K_m^* : L^1(\partial D) \rightarrow L^1(\partial D)$  is bounded. By (8.2), it is easily find that  $K_m^* : L^\infty(\partial D) \rightarrow L^\infty(\partial D)$  is also bounded. Then by the interpolation of operators,  $K_m^* : L^p(\partial D) \rightarrow L^p(\partial D)$  is bounded for  $1 < p < \infty$ .

Next turn to (8.18) under (8.17). By the definition of the operator  $K_m^*$  and Theorem 5.4, we have

$$(8.20) \quad \begin{aligned} \int_{\partial D} K_m^* f(P) d\sigma(P) &= \int_{\partial D} \left[ \int_{\partial D} \langle D_m(Q, P), n_P \rangle f(Q) d\sigma(Q) \right] d\sigma(P) \\ &= \int_{\partial D} \left[ \int_{\partial D} \langle D_m(Q, P), n_P \rangle d\sigma(P) \right] f(Q) d\sigma(Q) \\ &= \int_{\partial D} \left[ \int_{\partial D} \langle \nabla \mathcal{D}_m(Q, P), n_P \rangle d\sigma(P) \right] f(Q) d\sigma(Q) \\ &= \int_{\partial D} \left[ \int_{\partial D} \frac{\partial}{\partial N_P} \mathcal{D}_m(Q, P) d\sigma(P) \right] f(Q) d\sigma(Q) \end{aligned}$$

where

(8.21)

$$\begin{aligned}
\int_{\partial D} \frac{\partial}{\partial N_p} \mathcal{D}_m(Q, P) d\sigma(P) &= \lim_{\epsilon \rightarrow 0} \int_{\partial D \setminus B(Q, \epsilon)} \frac{\partial}{\partial N_p} \mathcal{D}_m(Q, P) d\sigma(P) \\
&= \lim_{\epsilon \rightarrow 0} \left( \int_{\partial D \setminus B(Q, \epsilon)} + \int_{\partial D \cap B(Q, \epsilon)} \right) \frac{\partial}{\partial N_p} \mathcal{D}_m(Q, P) d\sigma(P) \\
&= \lim_{\epsilon \rightarrow 0} \int_{D \setminus B(Q, \epsilon)} \operatorname{div} \nabla (\mathcal{D}_m(Q, X)) dX \\
&= \int_D \Delta \mathcal{D}_m(Q, X) dX \\
&= \int_D \mathcal{D}_{m-1}(Q, X) dX \\
&= \mathcal{N}_{m-1}(Q)
\end{aligned}$$

in which Gauass's divergence theorem, and the following easy facts are used (by Lebesgue's dominated convergence theorem, the details are similar to the argument of Lemma 8.2):

$$(8.22) \quad \lim_{\epsilon \rightarrow 0} \int_{\partial D \cap B(Q, \epsilon)} \frac{\partial}{\partial N_p} \mathcal{D}_m(Q, P) d\sigma(P) = 0$$

and

$$\begin{aligned}
(8.23) \quad \lim_{\epsilon \rightarrow 0} \int_{D \setminus B(Q, \epsilon)} \operatorname{div} \nabla (\mathcal{D}_m(Q, X)) dX &= \lim_{\epsilon \rightarrow 0} \int_{D \setminus B(Q, \epsilon)} \Delta \mathcal{D}_m(Q, X) dX \\
&= \int_D \Delta \mathcal{D}_m(Q, X) dX.
\end{aligned}$$

Therefore, by (8.17), (8.20) and (8.21), we have

$$\int_{\partial D} K_m^* f(P) d\sigma(P) = \int_{\partial D} \mathcal{N}_{m-1}(Q) f(Q) d\sigma(Q) = 0.$$

□

**Theorem 8.12.** *Let  $D$  be a bounded Lipschitz domain, and  $K_m$ ,  $m \geq 2$  be the same as in Theorem 3.5, then  $K_m : L^p(\partial D) \rightarrow L^p(\partial D)$  is bounded for  $1 \leq p \leq \infty$ .*

*Proof.* By duality in term of Theorem 8.10, or directly verify by a similar argument to Theorem 8.10 by invoking Lemma 8.2. □

**Theorem 8.13.** *Let  $D$  be a bounded Lipschitz domain, and  $M_j$ ,  $j \geq 2$  be the  $j$ th layer  $\mathcal{D}$ -potential, then  $M_j : L^p(\partial D) \rightarrow L^p(D)$  is bounded for  $1 \leq p \leq \infty$ .*

*Proof.* By Lemma 8.3 and the Riesz-Thorin interpolation theorem of operators, it is similar to Theorem 8.10. □

**Theorem 8.14.** *Let  $D$  be a bounded Lipschitz domain, and  $\mathcal{M}_j$ ,  $j \geq 1$  be the  $j$ th layer  $\mathcal{S}$ -potential, then  $\mathcal{M}_j : L^p(\partial D) \rightarrow L^p(\partial D)$  is bounded for  $1 \leq p \leq \infty$ .*

*Proof.* It is similar to Theorem 8.10 by using Lemma 8.5, the claims in Remark 8.9 and the interpolation of operators. □



**Theorem 8.15.** *Let  $D$  be a bounded Lipschitz domain, and  $\mathcal{M}_j$ ,  $j \geq 1$  be the  $j$ th layer  $\mathcal{S}$ -potential, then  $\mathcal{M}_j : L^p(\partial D) \rightarrow L^p(D)$  is bounded for  $1 \leq p \leq \infty$ .*

*Proof.* It is similar to Theorem 8.10 by using Lemma 8.6, the claims in Remark 8.9 and the interpolation of operators.  $\square$

**Theorem 8.16.** *Let  $D$  be a bounded Lipschitz domain, and  $\mathcal{M}_j$ ,  $j \geq 2$  be the  $j$ th layer  $\mathcal{S}$ -potential, then  $\nabla \mathcal{M}_m : L^p(\partial D) \rightarrow L^p(D)$  is bounded for  $1 \leq p \leq \infty$ .*

*Proof.* It is similar to Theorem 8.10 by using Lemma 8.8 and the interpolation of operators.  $\square$

*Remark 8.17.* By Lemma 8.8 and the statements in Remarks 8.4, 8.7 and 8.9, in fact, by performing a similar argument to Theorem 8.10, we have that all the operators  $\mathcal{M}_j$  and  $\nabla \mathcal{M}_j$  are bounded from  $L^p(\partial D)$  to  $L^p(\overline{D})$  for any  $j \geq 2$  and  $1 \leq p \leq \infty$ , whereas  $\mathcal{M}_j : L^p(\partial D) \rightarrow L^p(\overline{D})$  is bounded for any  $j \geq 1$  and  $1 \leq p \leq \infty$ .

The following lemma is crucial to the non-tangential maximal estimates of solutions for the  $L^p$  polyharmonic BVPs discussing in this section, whose analogue is also significant to the corresponding estimates of the Dirichlet and Neumann problems in  $L^p$  for Laplace's equation (see [11, 12]).

**Theorem 8.18.** *Let  $D$  be a bounded Lipschitz domain with the coordinate systems  $(L_j, Q_j)$ ,  $\varphi_j$  and  $\pi_j$  as the same as in the proof of Lemma 8.2,  $M_m$ ,  $m \geq 1$  be the  $j$ th layer  $\mathcal{D}$ -potential. If  $X \in L_{j_0} \cap D$  for some  $1 \leq j_0 \leq s$ , set  $P \in \partial D \cap L_{j_0}$  with  $\pi_{j_0}(X) = \pi_{j_0}(P)$ , and  $\rho = |X - P|$ , then for any  $f \in L^{p_m}(\partial D)$ ,*

$$(8.24) \quad |M_m f(X) - (K_m)_\rho f(P)| \leq C M^* f(P)$$

where

$$(8.25) \quad (K_m)_\rho f(P) = \int_{\partial D \setminus B_\rho(P)} \langle D_m(P, Q), n_Q \rangle f(Q) d\sigma(Q),$$

the maximal function  $M^* f$  is defined as follows

$$(8.26) \quad M^* f(P) = \sup_{r>0} \left[ \frac{1}{\sigma(\partial D \cap B_r(P))} \int_{\partial D \cap B_r(P)} |f(Q)| d\sigma(Q) \right], \quad P \in \partial D$$

and

$$(8.27) \quad p_m \in \begin{cases} (1, \infty), & m = 1; \\ [1, \infty], & m \geq 2. \end{cases}$$

*Proof.* It is due to Dahlberg in the case of  $m = 1$  (i.e., Proposition 1.1, [11]). To other cases, as the proof of Lemma 8.2, by invoking the local coordinates, it can be attained by a similar argument to Dahlberg's one.  $\square$

**Theorem 8.19.** *Let  $D$  be a bounded Lipschitz domain,  $(K_m)_\rho$  be defined as (8.25). For any  $f \in L^{p_m}(\partial D)$ , define the maximal operator*

$$(8.28) \quad K_m^\# f(P) = \sup_{\rho>0} |(K_m)_\rho f(P)|, \quad P \in \partial D,$$

then

$$(8.29) \quad \|K_m^\# f\|_{L^{p_m}(\partial D)} \leq C \|f\|_{L^{p_m}(\partial D)},$$

where  $p_m$  is given by (8.27), and  $C$  is a constant depending only on  $m, n, p_m$  and  $D$ .

*Proof.* The case of  $m = 1$  is a deep and classical result [11, 28, 50]. By Lemma 8.2 and the interpolation of operators, other cases follows.  $\square$

**Theorem 8.20.** *Let  $D$  be a bounded Lipschitz domain,  $M_m$ ,  $m \geq 1$  be the  $j$ th layer  $\mathcal{D}$ -potential, then for any  $f \in L^p(\partial D)$  with  $1 < p < \infty$ ,*

$$(8.30) \quad \|M(M_m f)\|_{L^p(\partial D)} \leq C \|f\|_{L^p(\partial D)},$$

where  $M(\cdot)$  is the nontangential maximal function given by (1.4), and  $C$  is a constant depending only on  $m, n, p$  and  $D$ .

*Proof.* Since  $M^* : L^p(\partial D) \rightarrow L^p(\partial D)$  is bounded for any  $1 < p < \infty$  (e.g., see [50]), then by Theorems 8.18 and 8.19, (8.30) follows immediately. The case of  $m = 1$  is classical.  $\square$

However, the multi-layer  $\mathcal{S}$ -potentials version of Lemmas 8.18-8.20 is the following

**Theorem 8.21.** *Let  $D$  be a bounded Lipschitz domain with the coordinate systems  $(L_j, Q_j)$ ,  $\varphi_j$  and  $\pi_j$  as the same as in the proof of Lemma 8.2,  $\mathcal{M}_m$ ,  $m \geq 1$  be the  $j$ th layer  $\mathcal{S}$ -potential. If  $X \in L_{j_0} \cap D$  for some  $1 \leq j_0 \leq s$ , set  $P \in \partial D \cap L_{j_0}$  with  $\pi_{j_0}(X) = \pi_{j_0}(P)$ , and  $\rho = |X - P|$ , then for any  $f \in L^{p_m}(\partial D)$ ,*

$$(8.31) \quad |\nabla \mathcal{M}_m f(X) - (\tilde{K}_m)_\rho f(P)| \leq C M^* f(P)$$

where

$$(8.32) \quad (\tilde{K}_m)_\rho f(P) = \int_{\partial D \setminus B_\rho(P)} \nabla \mathcal{D}_m(P, Q) f(Q) d\sigma(Q),$$

the maximal function  $M^* f$  are defined by (8.26),  $\nabla$  is the gradient operator and  $p_m$  is given by (8.27).

*Proof.* It is similar to Theorem 8.18.  $\square$

**Theorem 8.22.** *Let  $D$  be a bounded Lipschitz domain,  $(\tilde{K}_m)_\rho$  be defined as (8.32). For any  $f \in L^{p_m}(\partial D)$ , set the maximal operator*

$$(8.33) \quad \tilde{K}_m^\# f(P) = \sup_{\rho > 0} |(\tilde{K}_m)_\rho f(P)|, \quad P \in \partial D,$$

then

$$(8.34) \quad \|\tilde{K}_m^\# f\|_{L^{p_m}(\partial D)} \leq C \|f\|_{L^{p_m}(\partial D)},$$

where  $p_m$  is given by (8.27), and  $C$  is a constant depending only on  $m, n, p_m$  and  $D$ .

*Proof.* Similar to Theorem 8.19.  $\square$

**Theorem 8.23.** *Let  $D$  be a bounded Lipschitz domain,  $\mathcal{M}_m$ ,  $m \geq 1$  be the  $j$ th layer  $\mathcal{S}$ -potential, then for any  $f \in L^p(\partial D)$  with  $1 < p < \infty$ ,*

$$(8.35) \quad \|M(\nabla \mathcal{M}_m f)\|_{L^p(\partial D)} \leq C \|f\|_{L^p(\partial D)},$$

where  $\nabla$  is the gradient operator,  $M(\cdot)$  is the nontangential maximal function given by (1.4), and  $C$  is a constant depending only on  $m, n, p$  and  $D$ .

*Proof.* Similar to Theorem 8.20.  $\square$

Now we can give the main results in this section as follows

**Theorem 8.24.** *Let  $\{K_m\}_{m=1}^\infty$  be the sequence of the Poisson fields, and  $D$  be a bounded Lipschitz domain in  $\mathbb{R}^{n+1}$  with boundary  $\partial D$ , then for any  $m > 1$ , there exists  $\varepsilon = \varepsilon(D) > 0$  such that the PHD problem (4.1) with the data  $f_j \in L^p(\partial D)$ ,  $2 - \varepsilon < p < \infty$ , is solvable and a solution is given by*

$$(8.36) \quad \begin{aligned} u(X) &= \sum_{j=1}^m \int_{\partial D} \langle K_j(X, Q), n_Q \rangle \tilde{f}_{j-1}(Q) d\sigma(Q), \\ &= \sum_{j=1}^m M_j \tilde{f}_{j-1}(X), \quad X \in D, \end{aligned}$$

where

$$(8.37) \quad \tilde{f}_{m-1} = \left( \frac{1}{2}I + T \right)^{-1} f_{m-1}$$

and

$$(8.38) \quad \tilde{f}_l = \left( \frac{1}{2}I + T \right)^{-1} \left( f_l - \sum_{j=l+2}^m K_{j-l} \tilde{f}_{j-1} \right)$$

with  $0 \leq l \leq m-2$ , which satisfying the following estimates

$$(8.39) \quad \|u - M_1 \tilde{f}_0\|_{L^p(D)} \leq C \sum_{j=1}^{m-1} \|f_j\|_{L^p(\partial D)}$$

and

$$(8.40) \quad \|M(u)\|_{L^p(\partial D)} \leq C \sum_{j=0}^{m-1} \|f_j\|_{L^p(\partial D)}$$

in which  $M(u)$  is the non-tangential maximal function of  $u$  on  $\partial D$ . Under any of the above two estimates, the solution (8.36) with (8.37) and (8.38) is unique.

*Proof.* It is similar to Theorem 4.5 by using Lemma 3.4, Theorems 8.12, 8.13 and 8.20.  $\square$

**Theorem 8.25.** *Let  $\{\mathcal{K}_m\}_{m=1}^\infty$  be the sequence of the polyharmonic fundamental solutions, and  $D$  be a bounded Lipschitz domain in  $\mathbb{R}^{n+1}$  with boundary  $\partial D$ , then for any  $m > 1$ , there exists  $\varepsilon = \varepsilon(D) > 0$  such that the PHN problem (6.1) with the data  $g_{m-1} \in L_0^p(\partial D)$ ,  $g_j \in L^p(\partial D)$ ,  $0 \leq j \leq m-2$ ,  $1 < p < 2 + \varepsilon$ , is solvable and a solution is given by*

$$(8.41) \quad \begin{aligned} u(X) &= \sum_{j=1}^m \int_{\partial D} \mathcal{K}_j(X, Q) \tilde{g}_{j-1}(Q) d\sigma(Q), \\ &= \sum_{j=1}^m \mathcal{M}_j \tilde{g}_{j-1}(X), \quad X \in D, \end{aligned}$$

where

$$(8.42) \quad \tilde{g}_{m-1} = \left( -\frac{1}{2}I + T^* \right)^{-1} g_{m-1}$$

and

$$(8.43) \quad \tilde{g}_l = \left( -\frac{1}{2}I + T^* \right)^{-1} \left( g_l + \sum_{j=l+2}^m K_{j-l}^* \tilde{g}_{j-1} \right)$$

with  $0 \leq l \leq m-2$ , which satisfying the following estimates

$$(8.44) \quad \|\nabla(u - \mathcal{M}_1 \tilde{g}_0)\|_{L^p(D)} \leq C \sum_{j=1}^{m-1} \|g_j\|_{L^p(\partial D)},$$

$$(8.45) \quad \|u\|_{L^p(D)} \leq C \sum_{j=0}^{m-1} \|g_j\|_{L^p(\partial D)}$$

and

$$(8.46) \quad \|M(\nabla u)\|_{L^p(\partial D)} \leq C \sum_{j=0}^{m-1} \|g_j\|_{L^p(\partial D)}$$

in which  $M(\nabla u)$  is the non-tangential maximal function of  $\nabla u$  on  $\partial D$ . The solution (8.41) with (8.42) and (8.43) is unique under (8.45), and unique up to a constant under (8.44) and (8.46).

*Proof.* It is similar to Theorem 6.7 by noting Remark 8.9 and using Lemmas 3.4 and 8.1, Theorems 8.10, 8.15, 8.16 and 8.23.  $\square$

*Remark 8.26.* By the second claim in Theorem 8.10, if

$$(8.47) \quad \int_{\partial D} \mathcal{N}_l \tilde{g}_j d\sigma = 0, \quad 1 \leq j \leq m-1 \text{ and } 1 \leq l \leq j,$$

where  $\mathcal{N}_l$  is the  $l$ th order Newtonian potential defined in (8.23), then

$$(8.48) \quad \int_{\partial D} K_{l+1}^* \tilde{g}_j d\sigma = 0, \quad 1 \leq j \leq m-1 \text{ and } 1 \leq l \leq j.$$

Therefore, by Lemma 8.1, (8.42) and (8.43), we obtain that  $\tilde{g}_j \in L_0^p(\partial D)$ , and further that  $g_j \in L_0^p(\partial D)$ ,  $0 \leq j \leq m-2$ .

**Theorem 8.27.** *Let  $\{\mathcal{K}_m\}_{m=1}^\infty$  be the sequence of the polyharmonic fundamental solutions, and  $D$  be a bounded Lipschitz domain in  $\mathbb{R}^{n+1}$  with boundary  $\partial D$ , then for any  $m > 1$ , there exists  $\varepsilon = \varepsilon(D) > 0$  such that the PHR problem (7.1) with the data  $h_j \in L_1^p(\partial D)$ ,  $0 \leq j < m$ ,  $1 < p < 2 + \varepsilon$ , is solvable and a solution is given by*

$$(8.49) \quad \begin{aligned} u(X) &= \sum_{j=1}^m \int_{\partial D} \mathcal{K}_j(X, Q) \tilde{h}_{j-1}(Q) d\sigma(Q), \\ &= \sum_{j=1}^m \mathcal{M}_j \tilde{h}_{j-1}(X), \quad X \in D, \end{aligned}$$

where

$$(8.50) \quad \tilde{h}_{m-1} = \mathcal{M}_1^{-1} h_{m-1}$$

and

$$(8.51) \quad \tilde{h}_l = \mathcal{M}_1^{-1} \left( h_l - \sum_{j=l+2}^m \mathcal{M}_{j-l} \tilde{h}_{j-1} \right)$$

with  $0 \leq l \leq m-2$ , which satisfying the following estimates

$$(8.52) \quad \|\nabla(u - \mathcal{M}_1 \tilde{h}_0)\|_{L^p(D)} \leq C \sum_{j=1}^{m-1} \|h_j\|_{L_1^p(\partial D)},$$

$$(8.53) \quad \|u\|_{L^p(D)} \leq C \sum_{j=0}^{m-1} \|h_j\|_{L_1^p(\partial D)}$$

and

$$(8.54) \quad \|M(\nabla u)\|_{L^p(\partial D)} \leq C \sum_{j=0}^{m-1} \|h_j\|_{L_1^p(\partial D)}$$

in which  $M(\nabla u)$  is the non-tangential maximal function of  $\nabla u$  on  $\partial D$ . The solution (8.49) with (8.50) and (8.51) is unique under (8.53), and unique up to a constant under (8.52) and (8.54).

*Proof.* It is similar to Theorem 7.2 by noting Remark 8.9 and invoking Lemma 7.1, Theorems 8.14-8.16, and 8.23.  $\square$

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